POSITROID LINKS AND BRAID VARIETIES

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Abstract. We study braid varieties and their relation to open positroid varieties. First, we construct a DG-algebra associated to certain braid words, possibly admitting negative crossings, show that its zeroth cohomology is an invariant under braid equivalence and positive Markov moves, and provide an explicit geometric model for its cohomology in terms of an affine variety and a set of locally nilpotent derivations. Second, we discuss four different types of braids associated to open positroid strata and show that their associated Legendrian links are all Legendrian isotopic. In particular, we prove that each open positroid stratum can be presented as the augmentation variety for different Legendrian fronts described in terms of either permutations, juggling patterns, cyclic rank matrices or Le diagrams. We also relate braid varieties to open Richardson varieties and brick manifolds, showing that the latter provide projective compactifications of braid varieties, with normal crossing divisors at infinity, and compatible stratifications. Finally, we state a conjecture on the existence and properties of cluster $\mathcal{A}$-structures on braid varieties.

Она любила Ричардсона
Не потому, чтобы прочла
А. С. Пушкин, *Евгений Онегин*[1]

1. Introduction

This article studies braid varieties [11, 62] and their relation to open positroid varieties [53]. In a nutshell, we study four braids associated to any open positroid variety, and develop new techniques to algebraically study their braid varieties, now including Markov moves and allowing for negative crossings. In addition, this paper brings to bear insight from contact and symplectic topology to explicitly study these braid varieties, with salient focus on the case of open positroid varieties in Grassmannians and the differential graded algebras of their associated Legendrian links.

An open positroid variety Π of the Grassmannian $\mathrm{Gr}(k,n)$ can be indexed by either of the following four pieces of data. First, a pair of permutations $u, w \in S_n$ such that $u \leq w$ in the Bruhat order and w is a k-Grassmannian permutation. Second, a k-bounded affine permutation $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ with period n. Third, a cyclic rank matrix r and, fourth, a Le diagram. The bijections between these objects and the description of their associated positroid varieties are provided in [53, 71]. In this article, we study four *braids*, one associated to each of these four pieces of data, and introduce their associated Legendrian links. The main geometric result of this manuscript is showing that these four Legendrian links are all Legendrian isotopic. This requires resolving two challenges: the dissonance in the number of strands between these braids and, more prominently, the necessity for introducing *negative* crossings in proving equivalences between braids (even between positive braids). The existence of negative crossings forbids us from simply using front projections to realize braid Legendrian isotopies, adding a layer of complexity due to the existence of negative degree Reeb chords. To our knowledge, the conceptual insight that certain Legendrian links, not just smooth links, underlie *each* of this three presentations of a positroid variety is also new.[2]

In turn, the main algebraic result is showing that the braid varieties associated to Δ-equivalent positive braids are $\mathbb{C}^*$-equivalent algebraic varieties, in the sense of [56]. For us, two braids are said to be Δ-*equivalent* if they are related by positive stabilizations, positive destabilizations, Reidemeister moves II and III, and conjugations of type $\beta\sigma_i \sim \sigma_{n-i}\beta$ for any $i \in [1, n]$, where β is n-stranded.[3] In

[1] *The reason she loved Richardson was not that she had read him* — A.S. Pushkin, *Eugene Onegin* (tr. V. Nabokov).

[2] In particular, the braid associated to (u, w) typically contains *negative crossings* and, prior to this work, braid varieties where only defined for *positive* braids words. The generalizations in this manuscript, which allow for negative crossings, will now relate the braid variety of a positroid braid for (u, w) with its positroid variety.

[3] Δ-equivalence of braids is imposed in this geometric context, rather than the standard notion of braid equivalence. Conjugations are forced to be $\beta\sigma_i \sim \sigma_{n-i}\beta$, instead of $\beta\sigma_i \sim \sigma_i\beta$, due to the necessary presence of a half-twist Δ.

particular, the two challenges above force us to develop new Markov moves for braid varieties and, crucially, introduce a new characterization of braid varieties associated to *any* braid word, possibly with *negative* crossings, which represents a given positive braid. (Indeed, Reidemeister II moves introduce *negative* crossings.) This new description requires the introduction of locally nilpotent derivations to capture a derived presentation of these algebraic varieties. This expands the foundational techniques available in the study of braid varieties – conceptually and pragmatically – and, as an immediate consequence of the geometric result above, gives a precise relation between some of the different viewpoints previously used to study open positroid varieties, including [33, 53, 77]. From the perspective of symplectic topology, we provide an algebraic closed formula for the differentials in the DG-algebra of Legendrian links obtained as the Lagrangian (-1)-closures of braid words and relate it to braid matrices and braid varieties.

The article also includes new results relating braid varieties to projective brick manifolds and open Richardson varieties. In particular, we show that brick varieties are good projective compactifications for our affine braid varieties, and we deduce the curious Lefschetz property for open Richardson varieties by combining our work and that of A. Mellit [62]. The article concludes with a discussion on conjectural matters regarding cluster $\mathcal{A}$-structures on braid varieties.

1.1. **Scientific Context.** Let Br_n be the braid group in n strands and $\mathrm{Br}_n^+ \subseteq \mathrm{Br}_n$ the monoid of positive braids. We denote by $\mathcal{B}_n$ the set of braid *words* on the n Artin generators $\sigma_1, \dots, \sigma_{n-1}$, of Br_n, and $\mathcal{B}_n^+$ the set of *positive* braid words. The set of braids Δ-equivalent to a braid word $\beta \in \mathcal{B}_n$ is denoted by $[\beta]$; we will write $[\beta] \in \mathrm{Br}_n$, implicitly understanding that any equivalence is a Δ-equivalence, i.e. our braid equivalences are taken to always be in the presence of a half-twist Δ. Now, let β be a positive braid word $\beta \in \mathcal{B}_n^+$, $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, and $\pi \in \mathrm{GL}(n, \mathbb{C})$ a permutation matrix. Associated to this braid word, we consider the *braid variety*

$$X(\beta; \pi) := \{(z_1, \dots, z_\ell) \ : \ B_\gamma(z_1, \dots, z_\ell)\pi \text{ is upper-triangular}\} \subseteq \mathbb{C}^\ell,$$

where the matrix $B_\gamma(z_1, \dots, z_\ell) \in \mathrm{GL}(n, \mathbb{C}[z_1, \dots, z_\ell])$ is defined to be the matrix product

$$B_\gamma(z_1, \dots, z_\ell) := B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell),$$

and the matrices $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ are defined by:

$$(B_i(z))_{jk} := \begin{cases} 1 & j = k \text{ and } j \neq i, i+1 \\ 1 & (j,k) = (i, i+1) \text{ or } (i+1, i) \\ z & j = k = i+1 \\ 0 & \text{otherwise;} \end{cases}, \quad \text{i.e. } B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

The matrices $B_i(z)$ are referred to as *braid matrices*, and the only non-trivial (2×2)-block is at the ith and $(i+1)$st rows. Braid varieties were introduced in [11, 62], where part of their geometry was studied in detail. In particular, we proved in [11] that $X(\beta_1; \pi) \cong X(\beta_2; \pi)$ if β_1 and β_2 are related by Reidemeister III moves; hence the name *braid* varieties. In the present article, the permutation (matrix) π will often be $\pi = w_{0,n} = [n, n-1, \dots, 1] \in S_n$ and we will often abbreviate $X(\beta)$ for $X(\beta; w_{0,n})$. Let $\Delta_n \in \mathcal{B}_n^+$ be a positive braid lift of the permutation $w_{0,n}$, i.e. Δ_n will be a braid word for the half-twist on n-strands; we abbreviate Δ_n by Δ if n can be implicitly understood by context. Braid varieties will have a prominent role in the results of this manuscript.

Significant focus will be devoted to certain braid words and positroid varieties, which we now discuss. These varieties first appeared in the study of total positivity [59, 60, 71, 73] and in the context of Poisson geometry [8]. Let $\Pi_{u,w}$ be the open positroid variety of the Grassmannian $\mathrm{Gr}(k, n)$ indexed by a pair of permutations $u, w \in S_n$, $u \leq w$ in Bruhat order and w a k-Grassmannian permutation. Consider the KLS-bijections between such pairs (u, w), bounded affine permutations $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, cyclic rank matrices r and Le-diagrams $\daleth$ established in [53, 71]. For instance, the bounded affine permutation $f(u, w) : \mathbb{Z} \longrightarrow \mathbb{Z}$ corresponding to a pair (u, w) is $f(u, w) := u^{-1} t_k w$, where t_k is the translation by the kth fundamental weight; conversely, f recovers (u, w). Here f is interpreted as an n-periodic $\mathbb{Z}$-bijection such that $i \leq f(i) \leq i + n$ for all $i \in \mathbb{Z}$. The four pieces of data (u, w), f, r and $\daleth$ are said to *represent the same positroid type* if they correspond to each other under these bijections. Each piece of data, (u,w), f, r, and Ⅎ, yields an open stratum $\Pi_{u,v}$, Π_f, Π_r, and $\Pi_{\text{Ⅎ}}$ in $\mathrm{Gr}(k,n)$, and $\Pi_{u,v}=\Pi_f=\Pi_r=\Pi_{\text{Ⅎ}}$ if (u,w), f, r, and Ⅎ represent the same positroid type, e.g. see [53].

In fact, each of these pieces of data, (u,w), f, r and Ⅎ also yields a *braid* word. These four braids, which we correspondingly denote $R_n(u,v)$, $J_k(f)$, $M_k(r)$ and $D_k(\text{Ⅎ})$, will be studied in detail in this article. A succinct description of these braids now follows; see also the detailed example in Subsection 1.3 below, where these braids are illustrated:

(i) **Richardson braid**. Let $u,w\in S_n$ be such that $u\leq w$ in the Bruhat order and w is a k-Grassmannian permutation. Consider the two n-stranded positive braids $\beta(u),\beta(w)\in \mathrm{Br}_n^+$ obtained by choosing reduced words for $u,w\in S_n$ and lifting the Coxeter generators of S_n to the Artin braid generators of Br_n. By definition, the Richardson braid associated to (u,w) is $R_n(u,w):=\beta(w)\cdot\beta(u)^{-1}$. This braid was recently introduced by P. Galashin and T. Lam in [33, 34].

(ii) **Juggling braid**. Given a bounded affine permutation $f:\mathbb{Z}\longrightarrow\mathbb{Z}$, which we assume to be of the form $f=f(u,w)$, consider the real plane $\mathbb{R}^2$ with Cartesian coordinates (x,y). We draw the integer values $1,2,\ldots,2n$ on the horizontal real axis $\{(x,y):y=0\}\subseteq\mathbb{R}^2$, and for each $i\in\mathbb{N}$, $1\leq i\leq n$, we draw the upper semicircumference arc A_i that starts at the point $(f(i),0)$ and ends at the point $(i,0)$. Since $i\leq f(i)$, the arcs move leftwards and it follows from [53] that under our assumptions on (u,w) there exist exactly k values of i such that $n<f(i)\leq 2n$. The arcs starting at such points correspond to the strands in the braid. Visually, the braid is just given by taking the union of all the arcs $A_i(f)$, $1\leq i\leq n$, declaring all the intersections $A_i\cap A_j$ to be positive crossings and locally smoothing (the endpoints of) any two incident arcs at an integer point i into a smooth segment. See Section 3 for details. By definition, the juggling braid $J_k(f)$ associated to $f=f(u,w)$ is this resulting positive k-stranded braid.

(iii) **Cyclic Rank Matrix braid**. Given a cyclic rank matrix $r=(r_{ij})$, we define the positive braid $M_k(r)$ by using the local models in Figure 1. Namely, for each entry r_{ij}, the adjacent entries must either be equal or differ by one. For each of the different possibilities near each entry, a certain piece of braid is drawn. Figure 1 depicts the models around an entry $r_{ij}=\rho$ with fixed value ρ: the piece of the braid is drawn in red. The cyclic rank matrix condition implies that these five models cover all possibilities. By definition, matrix braid $M_k(r)$ associated to the cyclic rank matrix r is this resulting positive braid. This braid was introduced in [77].

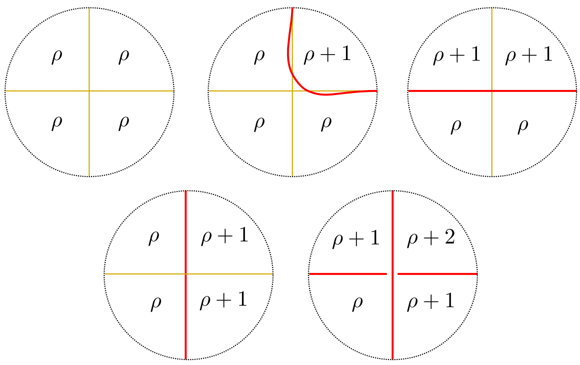


FIGURE 1. The local models for the k-stranded braid $M_k(r)$ associated to a cyclic rank matrix $r=(r_{ij})$ near each entry of the matrix. In these local models, the value of a given entry r_{ij} has been set to $r_{ij}=\rho$ and the braid is depicted in red. The yellow lines just indicate the separation between the matrix entries of r.

(iv) **Diagram braid**. Given u,w as above, one can associate [71] a Young diagram to w and a certain subset of its cells, known as Le-diagram $\text{Ⅎ}(u,w)$, to u. Given such a diagram, we construct a certain k-strand braid word $D_k(\text{Ⅎ})$, which in general is not positive. Each column in the Le-diagram Ⅎ corresponds to a certain conjugate of an interval braid; see Figure 2 for an example. The precise definition is in Section 3, but see also the example in Subsection 1.3 below for a sneak peek.

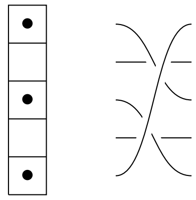


FIGURE 2. A column in a Le diagram $\text{Ꞁ}(u,w)$ and the corresponding braid.

Either of these four braids will be referred to as a *positroid braid*. Note that the n-stranded braid $R_n(u,w)$ is *not* necessarily a positive braid and neither is the k-stranded braid $D_k(\text{Ꞁ})$, whereas the braids $J_k(f), M_k(r)$ are positive. To our knowledge, the juggling and diagram braids are new; they certainly both have a crucial role in our arguments. In addition, a novel perspective for the conceptual understanding of all these positroid braids is given by considering the following associated *Legendrian links*:

Definition 1.1. The Legendrian link $\Lambda(u,w) \subseteq (\mathbb{R}^3,\xi)$ associated to (u,w) is the Legendrian lift of the Lagrangian (-1)-closure of the braid $R_n(u,w)\Delta_n^2$. The Legendrian link $\Lambda(f) \subseteq (\mathbb{R}^3,\xi)$ associated to f is the Legendrian (-1)-closure of the braid $J_k(f)\Delta_k$, and the Legendrian link $\Lambda(r) \subseteq (\mathbb{R}^3,\xi)$ associated to r the Legendrian (-1)-closure of the braid $M_k(r)$. Finally, the Legendrian link $\Lambda(\text{Ꞁ}) \subseteq (\mathbb{R}^3,\xi)$ is the Legendrian lift of the Lagrangian (-1)-closure of the braid $D_k(\text{Ꞁ})\Delta_k^2$. □

Definition 1.1 is new and, we believe, gives the correct Legendrian links to be considered. Indeed, a suitable augmentation variety associated to these Legendrian links precisely coincides with the corresponding open positroid variety. Given that, either of these four types of Legendrian links will be referred to as *positroid links*. Figure 3 exhibits three of these four links in the Lagrangian projection. The reader is referred to Figure 14 for a table of the possible closures in the front and Lagrangian projections, and Section 2 for more details. The precise engineering of which Δ_k or Δ_n factor is optimal in Definition 1.1 has been guided by our upcoming Theorem 1.3.

Remark 1.2. Note that $R_n(u,w)\Delta_n^2 \in \mathcal{B}_n$ is *not* a positive braid word, despite representing a positive braid $[R_n(u,w)\Delta_n^2] \in \mathrm{Br}_n^+$. This prevents us from naively describing $\Lambda(u,w) \subseteq (\mathbb{R}^3,\xi_{\mathrm{st}})$ with a braid word in a Legendrian front projection, and thus our need to employ a Lagrangian projection. Indeed, in a front projection *all* crossings of a Legendrian front must be positive crossings. In contrast, Lagrangian projections of Legendrian links allow for negative crossings. □

1.2. **Main Results.** The manuscript contains two main new results: Theorem 1.3, of a geometric nature, and Theorem 1.5, of an algebraic nature. En route, we also prove results, such as Theorem 1.8 relating the affine braid varieties $X(\beta)$ to L. Escobar's projective *brick varieties* $\mathrm{brick}(\beta)$ [20], showing that the latter – which depends on the choice of braid word – is a smooth projective good compactification of the former, which is a smooth affine variety. Section 2 will also contain a comprehensive discussion on Legendrian links and a proof that braid pairs yield quotients isomorphic to the zeroth cohomology group of the Legendrian contact DG-algebra.

First, Theorem 1.3 reads as follows:

Theorem 1.3. *Let $u,w \in S_n$ be such that $u \le w$ in the Bruhat order and w is a k-Grassmannian permutation, f a bounded affine permutation, r a cyclic rank matrix, and Ꞁ a Le-diagram. Suppose that these four pieces of data represent the same positroid type. Then*

(i) The n-stranded braid $R_n(u,w)$ and the k-stranded braid $J_k(f)\Delta_k^{-1}$ are equivalent, up to adding unlinked disjoint strands. Also, the n-stranded braid $R_n(u,w)$ and the k-stranded braid $D_k(\text{Ꞁ})$ are equivalent, up to adding unlinked disjoint strands.

(ii) The k-stranded positive braids $J_k(f)\Delta_k$ and $M_k(r)$ are conjugate, and thus equivalent.

In particular, the four Legendrian positroid links $\Lambda(u,w),\Lambda(f),\Lambda(r),\Lambda(\text{Ɉ}) \subseteq (\mathbb{R}^3,\xi_{st})$ are Legendrian isotopic, up to unlinked max-tb Legendrian unknots, and thus their associated DG-algebras are stable tame isomorphic. □

Theorem 1.3.(i) is the main intricate statement, as it relates braids, such as $R_n(u,w)$ and $J_k(f)\Delta_k^{-1}$, in a different number of strands and typically with negative crossings. We will explicitly prove this first equivalence between $R_n(u,w)$ and $J_k(f)\Delta_k^{-1}$ by using the Le-braid $D(\text{Ɉ})$, and comparing both the Richardson braid and the juggling braid to $D(\text{Ɉ})$; this will occupy the majority of Section 3. Figure 3 depicts the Legendrian isotopies between the positroid Legendrian links implied by Theorem 1.3. These are drawn in the Lagrangian projection; see Section 2 for a discussion on Legendrian fronts and Lagrangian projections for Legendrian links in standard contact $(\mathbb{R}^3,\xi_{\text{st}})$.

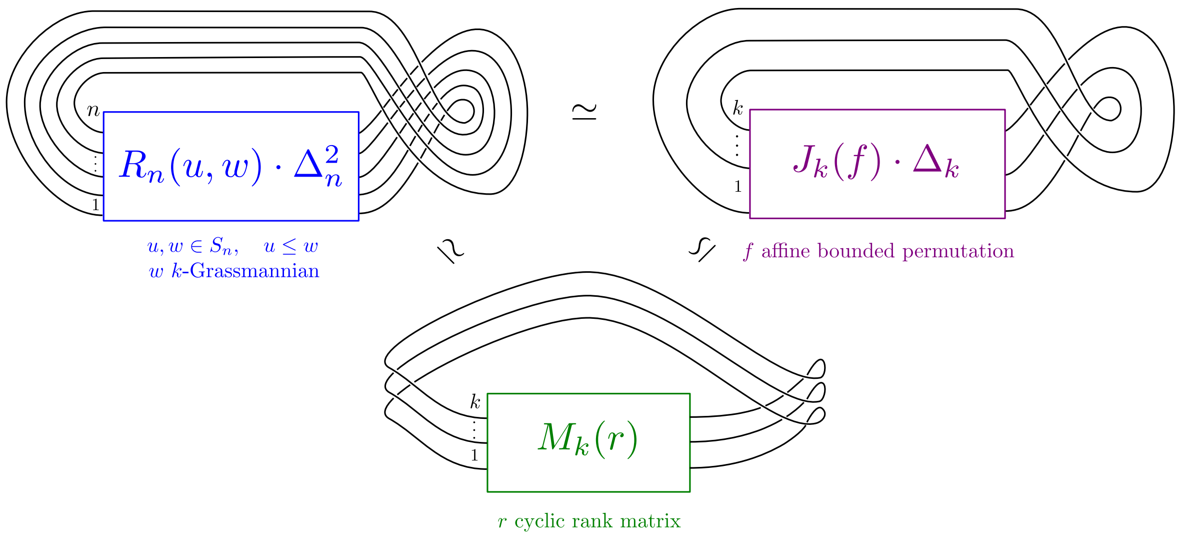


FIGURE 3. The Legendrian isotopies for the three main Legendrian links underlying the different presentations of a fixed open positroid variety in $\text{Gr}(k,n)$. The Richardson braid $R_n(u,w)$, associated to a pair of n-permutations (u,w), is n-stranded, as depicted in the upper-left. The juggling braid $J_k(f)$, associated to a bounded affine permutation f, is k-stranded, as depicted in the upper-right. The matrix braid $M_k(r)$, associated to a cyclic rank matrix, is k-stranded, as depicted in the second row.

Second, we now focus on the braid variety $X(\beta;w_{0,\ell})$, where $\beta\in\mathcal{B}_\ell^+$ is an ℓ-stranded positive braid word. Suppose that β is Δ-equivalent to an m-stranded braid word $\vartheta\in\mathcal{B}_m$, where ϑ might now have negative crossings. That is, ϑ is obtained from β by applying a sequence of positive stabilizations, positive destabilizations, Δ-conjugations and Reidemeister moves II and III. We would like to be able to define the braid variety $X(\vartheta;w_{0,m})$ and compare it to $X(\beta;w_{0,\ell})$. The main issue is that $X(\eta;\pi)$ is, so far, only defined for η a *positive* braid word. The dissonance in the number of strands $\ell,m\in\mathbb{N}$ also needs to be addressed, even in the context of positive braids words $\beta_1\in\mathcal{B}_\ell^+,\beta_2\in\mathcal{B}_m^+$ which are connected through positive stabilizations and destabilizations. That is, we also need to compare the braid varieties $X(\beta_1;w_{0,\ell})$ and $X(\beta_2;w_{0,m})$, i.e. study the behavior braid varieties under positive Markov moves. This will be addressed in Section 2.

Remark 1.4. Even if we are *only* interested in comparing two positive braid words $\beta_1\in\mathcal{B}_\ell^+,\beta_2\in\mathcal{B}_m^+$ which yield the same smooth positive link, one *must* typically encounter braid words with negative crossings in order to go between the two positive braid words β_1 and β_2. □

In order to address the appearance of negative crossings, we introduce a new pair $(X(\eta),V(\eta))$ for *any* braid word $\eta\in\mathcal{B}_m$ such that $[\eta]\in\text{Br}_m^+$. Here $V(\eta)$ is a set of locally nilpotent derivations on (the coordinate ring of) an affine variety $X(\eta)$. In a nutshell, the algebraic variety $X(\eta)$ comes from the positive crossings of $\eta\Delta_m$, as in the braid variety case, and each negative crossing of η contributes with a $\mathbb{C}$-action on $X(\eta)$. In particular, $X(\eta) = X(\eta; w_{0,m})$ and $V(\eta) = \emptyset$ if η is a positive braid word. In general, if we only have that $\eta\Delta_m$ is *equivalent* to a positive braid word $\beta \in \mathcal{B}_\ell^+$, we will prove that the set of locally nilpotent derivations actually integrates to a free action whose quotient $X(\eta)/V(\eta)$ is algebraically isomorphic to the braid variety $X(\beta; w_{0,\ell})$. In summary, we prove:

Theorem 1.5. *Let $\vartheta_\ell \in \mathcal{B}_\ell$, $\vartheta_m \in \mathcal{B}_m$, $\ell, m \in \mathbb{N}$, $m \leq \ell$, be two Δ-equivalent braid words such that $[\vartheta_\ell] \in \mathrm{Br}_\ell^+$ and $[\vartheta_m] \in \mathrm{Br}_m^+$. The following holds:*

(i) Suppose that there exists a Δ-equivalence with no stabilizations. Then there exists an affine algebraic isomorphism

$$X(\vartheta_\ell)/V(\vartheta_\ell) \cong X(\vartheta_m)/V(\vartheta_m),$$

where the locally nilpotent derivations $V(\vartheta_\ell), V(\vartheta_m)$ integrate to free actions on $X(\vartheta_\ell), X(\vartheta_m)$ and their quotients are smooth affine algebraic varieties.

(ii) Suppose that $\vartheta_\ell = \eta_\ell\Delta_\ell$, $\vartheta_m = \eta_m\Delta_m$ for some equivalent braid words $\eta_\ell \in \mathcal{B}_\ell, \eta_m \in \mathcal{B}_m$. Then $X(\vartheta_\ell)/V(\vartheta_\ell)$ and $X(\vartheta_m)/V(\vartheta_m)$ are $\mathbb{C}^*$-equivalent. That is, there exist $d_1, d_2 \in \mathbb{N}$ and an algebraic isomorphism

$$X(\vartheta_\ell)/V(\vartheta_\ell) \times (\mathbb{C}^*)^{d_1} \cong X(\vartheta_m)/V(\vartheta_m) \times (\mathbb{C}^*)^{d_2},$$

of affine varieties, where the locally nilpotent derivations $V(\vartheta_\ell), V(\vartheta_m)$ integrate to free actions on $X(\vartheta_\ell), X(\vartheta_m)$ and their quotients are smooth affine algebraic varieties.

In particular, there exist $d_1, d_2 \in \mathbb{N}$ and an algebraic isomorphism

$$X(\beta_\ell\Delta_\ell; w_{0,\ell}) \times (\mathbb{C}^*)^{d_1} \cong X(\beta_m\Delta_m; w_{0,m}) \times (\mathbb{C}^*)^{d_2}.$$

between the affine braid varieties of any two equivalent positive words $\beta_\ell\Delta_\ell \in \mathcal{B}_\ell^+, \beta_m\Delta_m \in \mathcal{B}_m^+$. □

Thus, the pair $(X(\vartheta), V(\vartheta))$ is the correct generalization of a braid variety $X(\beta; w_0)$ when a braid word, equivalent to a positive braid, acquires negative crossings. Hence, we can now compare braid varieties for positive braid words which are equivalent, even if negative crossings are introduced when realizing an equivalence. Note that the hypothesis in Theorem 1.5.(ii) is satisfied in all the cases we consider, including that of the braid word $R_n(u, w)\Delta_n$ for the Richardson positroid link. In fact, we will prove Theorem 1.5 by explicitly understanding the change of the pair $(X(\vartheta), V(\vartheta))$ under each operation:

(i) A Reidemeister II move that creates a canceling pair of crossings changes the affine variety $X(\vartheta)$ by adding a trivial $\mathbb{C}$-factor and adding an additional locally nilpotent derivation to $V(\vartheta)$. This is done such that the quotient $X(\vartheta)/V(\vartheta)$ remains invariant.

(ii) Either type of Reidemeister III moves and Δ-conjugations in ϑ do not change the affine variety $X(\vartheta)$ nor the action of $V(\vartheta)$ up to isomorphism.

(iii) Positive stabilization and destabilization change the affine variety $X(\vartheta)$ by adding or subtracting a trivial $\mathbb{C}^*$-factor, and the isomorphism respects the action of the locally nilpotent derivations $V(\vartheta)$. Note that the hypothesis that ϑ is of the form $\vartheta = \eta\Delta$ is already imposed at the smooth level, since the closures we consider are all (-1)-framed and thus the standard Markov stabilization needs to be modified by inserting a factor of Δ.

Remark 1.6. Geometrically, the locally nilpotent derivations $V(\eta)$ are given by a count of certain regions in the braid word η, as we will explain; this insight comes from considering negative crossings as (-1)-graded Reeb chords in the Lagrangian projection and counting pseudo-holomorphic strips. In the same vein, the stabilizing $\mathbb{C}^*$-factor that appears in Theorem 1.5 parallels the fact that the Legendrian DG-algebra is only invariant up to stable tame isomorphism. The technical hypothesis $[\vartheta_\ell] \in \mathrm{Br}_\ell^+$ and $[\vartheta_m] \in \mathrm{Br}_m^+$ in Theorem 1.5 is related to the concept of braid admissibility for a Lagrangian projection, as introduced in [12]; this hypothesis is met in all interesting examples through the manuscript, including the cases of positroid braids. See Subsection 2.5 for more details on contact topology. □

We shall also show that the pair $(X(\vartheta), V(\vartheta))$ provides a geometric model for computing the zeroth cohomology group of a DG-algebra associated to certain braid words, see Section 2. This DG-algebra will be constructed in Subsection 2.3 entirely using braid matrices associated to the braid words. It will be later shown to be DG-isomorphic to a Floer theoretic invariant associated to Legendrian links, in Subsection 2.5. In particular, we will show that the augmentation varieties of the Legendrian positroid links are isomorphic to the corresponding open positroid stratum, and similarly with their lifts to Richardson varieties.

We will combine Theorems 1.3 and 1.5, together with results of Knutson-Lam-Speyer [53] to show that a positroid variety $\Pi_{u,w}$ in the Grassmannian $\mathrm{Gr}(k,n)$ can be expressed in terms of braid varieties: either using the n-stranded braid $R_n(u,w)$, which is not necessarily positive; or the k-stranded braid $J_k(f)$, which is always positive. For instance, we prove the following result.

Theorem 1.7. *Let $u, w \in S_n$ with $u \leq w$ in Bruhat order, w a k-Grassmannian permutation, and $f := u^{-1}t_k w$ the corresponding k-bounded affine permutation. Then we have isomorphisms*

$$\Pi_{u,w} \cong X(R_n(u,w)\Delta_n)/V \cong X(\beta(w)\beta(u^{-1}w_{0,n}); w_{0,n}) \cong X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^{n-s-k}$$

of affine algebraic varieties, where $s := \#\{i \in [1,n] : f(i) = i\}$ is the number of fixed points of f. $\square$

Third, we relate braid varieties $X(\beta; w_0)$ to intersections of Schubert cells and brick manifolds. In her study of subword complexes, L. Escobar introduced a class of algebraic varieties called *brick manifolds*, see [20, Definition 3.2]. They are parametrized by a positive braid word β, denoted Q in [20], containing w_0 as a subword; she proved in [20, Theorem 3.3] that brick manifolds are smooth and projective. She also found certain explicit stratifications of brick manifolds, where the dual complexes of spherical subword complexes of type A, as introduced by A. Knutson and E. Miller, describe the adjacency of strata. We denote the brick manifold of β by $\mathrm{brick}(\beta)$ and its maximal open stratum in this stratification by $\mathrm{brick}^\circ(\beta)$. Let us focus on brick manifolds with Lie group $G = \mathrm{SL}_n(C)$, where the corresponding Weyl group is S_n and the associated braids are indeed elements of the Artin braid group Br_n.

Given a positive braid word $\beta \in \mathcal{B}^+$ which contains w_0 as a subword we consider its brick manifold $\mathrm{brick}(\beta)$. Note that $\mathrm{brick}(\beta)$ depends on the choice of braid word $\beta \in \mathcal{B}^+$, and not just the braid element $[\beta] \in \mathrm{Br}^+$, whereas $X(\beta; w_0)$ only depends on the positive braid $[\beta]$. The precise relation we establish between braid varieties and brick manifolds is the following:

Theorem 1.8. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ be a positive braid word, $\reflectbox{$\beta$} \in \mathcal{B}_n$ its opposite braid word, $\delta(\beta)$ the Demazure product of β, and consider the truncations $\beta_j := \sigma_{i_1} \cdots \sigma_{i_j}$, $j \in [1, \ell]$. The following holds:*

(*i*) *The algebraic map*

$$\Theta : \mathbb{C}^\ell \longrightarrow \mathscr{F}\ell_n^{\ell+1}, \quad (z_1, \ldots, z_\ell) \mapsto (\mathscr{F}^{st}, \mathscr{F}^1, \ldots, \mathscr{F}^\ell),$$

where $\mathscr{F}^j$ is the flag associated to the matrix $B^{-1}_{\reflectbox{$\beta$}_j}(z_{\ell-j+1}, \ldots, z_\ell)$, restricts to an isomorphism

$$\Theta : X(\reflectbox{β}; \delta(\beta)) \xrightarrow{\cong} \mathrm{brick}^\circ(\beta),$$

of affine varieties.

(*ii*) *The complement to $X(\reflectbox{$\beta$}; \delta(\beta))$ in* $\mathrm{brick}(\beta)$ *is a normal crossing divisor. Its components correspond to all possible ways to remove a letter from $\reflectbox{$\beta$}$ while preserving its Demazure product.*

In particular, $\mathrm{brick}(\beta)$ *is a smooth projective good compactification of the affine variety $X(\reflectbox{$\beta$}; \delta(\beta))$.*

Theorem 1.8, combined with [20], clarifies the connection between braid varieties and the combinatorics of subword complexes. This allows us to translate properties of spherical subword complexes via brick manifolds to braid varieties, and vice versa. In particular, the *rotation invariance* of the subword complex for (β, w_0) proved in [14, Proposition 3.9] can now be interpreted via invariance of $\mathrm{brick}(\beta)$ and of $X(\beta; w_0)$ under conjugations of $\gamma = \beta\Delta^{-1}$ by simple reflections.

Remark 1.9. Note that a consequence of Theorem 1.8 (i) is that the variety $X(\reflectbox{$\beta$}; \delta(\beta))$ is smooth, since L. Escobar proved in [20, Theorem 3.3] that brick varieties are smooth. In particular, $X(\beta; w_0)$ is smooth whenever $\delta(\beta) = w_0$. $\square$

Finally, the article concludes with a discussion on conjectural matters regarding the existence of cluster $\mathcal{A}$-structures on the coordinate rings of braid varieties and their properties.

For instance, open brick manifolds $\mathrm{brick}^\circ(\beta)$ are subvarieties in open Bott-Samelson varieties and, if β contains Δ as a subword, the corresponding brick manifold is a half decorated double Bott-Samelson cell, as introduced in [35, 76]. These cells are endowed with cluster $\mathcal{A}$-structures and thus Theorem 1.8 implies that certain braid varieties are cluster $\mathcal{A}$-varieties. In fact, there are more classes of braid varieties whose coordinate rings are known to admit (upper) cluster algebra structures; e.g. P. Galashin and T. Lam proved in [32] that the coordinate rings of all open positroid varieties admit cluster algebra structures. More generally, for all open Richardson varieties, not necessarily in type A, the proof of the existence of upper cluster algebra structures was recently announced by P. Cao and B. Keller [9], based on works of B. Leclerc [57] and E. Ménard [63]; and in type A, it was proved independently by G. Ingermanson [48]. Based on the results about cluster structures listed above and the discussion on Section 5, we will present a series of conjectural results in Conjecture 5.1. A first part of Conjecture 5.1 reads:

Conjecture 1.10. *The coordinate ring of any braid variety $X(\eta)$ admits a structure of a cluster algebra. The exchange type of the mutable part of its defining quiver is preserved under Reidemeister II moves, Reidemeister III moves and Δ-conjugations of the braid word η. In addition, each such move gives rise to a quasi-cluster transformation. A positive stabilization adds one frozen vertex to the defining quiver, and a positive destabilization specializes one frozen variable to 1.* $\square$

See Section 5 and Conjecture 5.1 for more details. For instance, we will state some stronger conjectural properties about such cluster structures, based on the combinatorial properties which we expect the corresponding quivers to enjoy. We will also explain the relation between the quivers considered in [32, 75] in the context of positroids, and the quivers considered in [35, 76] in the context of half decorated double Bott-Samelson cells.

As a side note, observe that positroid varieties are often considered inside Grassmannian varieties, whereas Richardson varieties are subvarieties in complete flag varieties, and Bott-Samelson cells live in Bott-Samelson varieties; as a consequence, the relation between these classes of varieties is, in a sense, somewhat hidden in the existing literature. The following Figure 4 is intended to clarify part of the interplay between them:

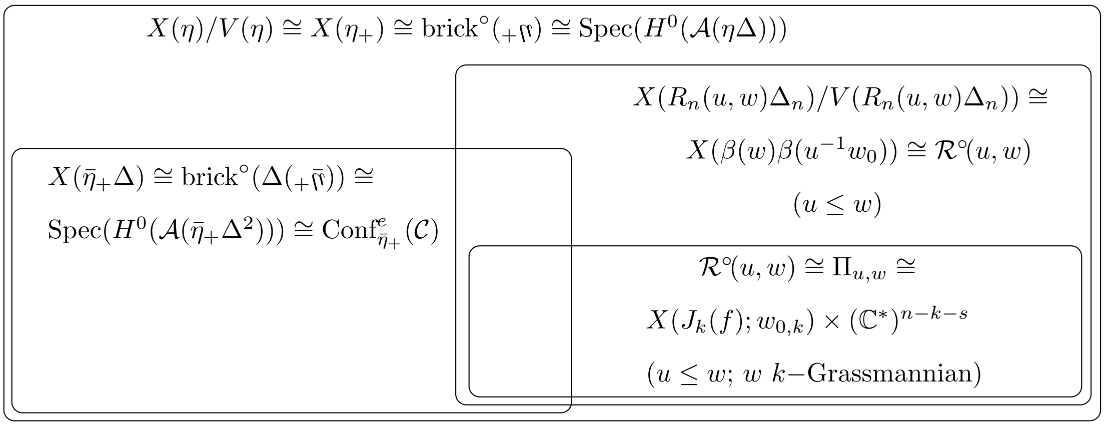


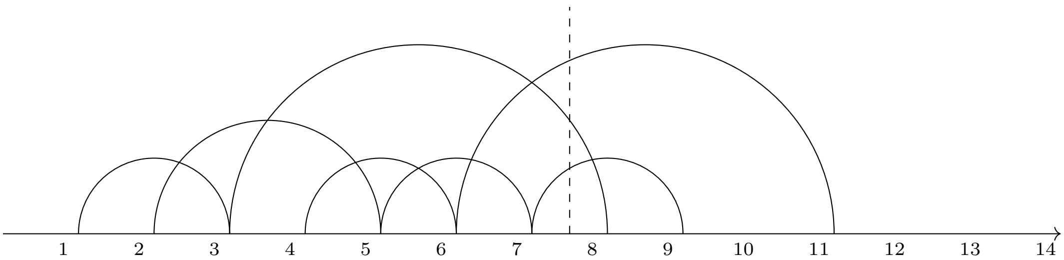
FIGURE 4. Some of the varieties appearing in the present article and in the literature [32, 35, 75]. Here $\eta_+, \bar\eta_+ \in \mathcal{B}_n^+$ are positive braid words, where η_+ represents $[\eta] \in \mathrm{Br}_n^+$. Braid varieties, which are open brick manifolds for $G = \mathrm{SL}_n$, are subsets of the open Bott-Samelson $\mathrm{OBS}(\eta\Delta)$. Open Richardson varieties are subsets of the flag variety $\mathscr{F}\ell_n \cong G/B$, and open positroid varieties are subsets of the Grassmannian $\mathrm{Gr}(k,n)$. The larger intersection in the middle consists of $\mathcal{R}^\circ(u,w)$ such that w admits a decomposition $w = vu$, with $l(w) = l(v) + l(u)$. The smaller intersection further requires the permutation w to be k-Grassmannian. The positroids in the small intersection were called skew Schubert varieties in [75].

Remark 1.11. Theorem 1.5 implies the existence of cluster structures on quotients of braid pairs $X(\eta)/V(\eta)$ for braids η that are related to those considered in [35], or related to Richardson braids via iterated positive (de)stabilizations, Δ-conjugations and Reidemeister moves II and III. Such varieties are related to Bott-Samelson cells and open Richardson varieties by products and quotients by algebraic tori; nevertheless, it appears to be an interesting open problem to describe all braids appearing in such way. For instance, the braid $\eta = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_3 \in \mathrm{Br}_4$ contains a negative crossing and it can be checked that it cannot be decomposed as a Richardson braid $R_4(u, w)$. However, it is related to a braid $\sigma_1^3\Delta_4$ by a sequence of Reidemeister II and III moves, and thus the quotient $X(\eta)/V(\eta)$ is isomorphic to $X(\sigma_1^3\Delta_4)$ and admits a cluster structure of type A_2. We will describe a class of braids appearing as juggling braids for opposite Schubert varieties in Grassmannians in Section 3.7. □

1.3. **A simple example.** We conclude this introduction with an explicit simple example, illustrating the different braids and links that feature in this manuscript. Let us choose $k = 3$ and $n = 7$, so that the positroid stratum will be a subset of the Grassmannian $\mathrm{Gr}(3, 7)$. Consider the following positroid data and associated braids:

(i) The pair of permutations (u, w) given by
$$w = [4, 5, 1, 6, 7, 2, 3] = s_3s_2s_1s_4s_3s_2s_5s_4s_6s_5 \in S_7,$$
$$u = [1, 3, 4, 2, 5, 6, 7] = s_2s_3 \in S_7,$$
where we are first using line notation and then $s_1, \dots, s_6$ are the simple transpositions generating S_7. Note that $u \leq w$ in the Bruhat order, and w is indeed 3-Grassmannian. Then, the 7-stranded Richardson braid $R_7(u, w)$ reads
$$R_7(u, w) := \beta(w)\beta(u)^{-1} = (\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_6\sigma_5)(\sigma_2\sigma_3)^{-1}.$$
Note that this Richardson braid is presented with a non-positive word $R_7(u, w) \notin \mathcal{B}_7^+$. In fact, it will not necessarily be the case that $[R_n(u, w)] \in \mathrm{Br}_n^+$, but only $[R_n(u, w)\Delta_n] \in \mathrm{Br}_n^+$, and even then the exact braid word associated to (u, w) contains negative crossings if u is non-empty.

(ii) The 3-bounded affine permutation $f = [3, 5, 8, 6, 7, 11, 9]$, where again we are using the notation that $f(i)$ is the ith entry of f, i.e. $f(1) = 3, f(2) = 5$ through $f(7) = 9$, and extended 7-periodically. This is the affine permutation associated to (u, w), as it is readily checked that $f = u^{-1}t_3w$ with (u, w) as in item (i) above. The arc diagram for this affine permutation is thus



and its associated 3-stranded braid word $J_3(f) = J_3([3, 5, 8, 6, 7, 11, 9])$ reads
$$J_3(f) = \sigma_2\sigma_1\sigma_2\sigma_2\sigma_2\sigma_1\sigma_1 \in \mathcal{B}_3.$$

(iii) The cyclic rank matrix $r = r(f)$ associated to $f = [3, 5, 8, 6, 7, 11, 9]$ is given by Figure 5 where we have marked the squares $(i, f(i))$ in gray, and highlighted the 3-stranded braid $M_3(r)$ in black, according to the rules in Figure 19. The braid reads
$$M_3(r) = \sigma_1\sigma_2\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1^2.$$

According to Theorem 1.3, the 7-stranded braid $[R_n(u, w)] \in \mathrm{Br}_7$ and $[J_3(f)\Delta_3^{-1}] \in \mathrm{Br}_3$ are equivalent. This can be verified in this instance via
$$J_3(f)\Delta_3^{-1} = \sigma_2\sigma_1\sigma_2^3\sigma_1^2 \cdot (\sigma_1\sigma_2\sigma_1)^{-1} = (\sigma_1\sigma_2\sigma_1)^{-1} \cdot \sigma_2\sigma_1\sigma_2^3\sigma_2^2 = \sigma_2^2\sigma_1^2,$$
$$R_7(u, w) = (\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_6\sigma_5)(\sigma_2\sigma_3)^{-1} = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_5(\sigma_2\sigma_3)^{-1} =$$

$i \backslash j$	1	2	3	4	5	6	7	8	9	10
1	1	2	2	3	3	3	3	3	3	3
2		1	2	3	3	3	3	3	3	3
3			1	2	3	3	3	3	3	3
4				1	2	2	2	3	3	3
5					1	2	2	3	3	3
6						1	2	3	3	3
7							1	2	2	2
8								1	2	2

FIGURE 5. Cyclic rank matrix for the introductory example.

$$= \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_4\sigma_4(\sigma_2\sigma_3)^{-1} = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_3\sigma_4\sigma_3\sigma_2(\sigma_2\sigma_3)^{-1} = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_3\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1} =$$
$$= \sigma_3\sigma_2\sigma_1\sigma_3\sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_2^{-1} = \sigma_2\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2^{-1}\sigma_3 = \sigma_2\sigma_3\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2^{-1} = \sigma_2^2\sigma_3^2 \simeq \sigma_2^2\sigma_1^2.$$

where the braid $R_7(u,w)$ has been simplified using conjugations, Reidemeister II and III moves, and Markov positive destabilizations. Theorem 1.5.(ii) states that the braid $[J_3(f)\Delta_3] \in \mathrm{Br}_3$ is equivalent to $[M_3(r)] \in \mathrm{Br}_3$. This is readily verified in this case as

$$J_3(f)\Delta_3 = \sigma_2\sigma_1\sigma_2^3\sigma_1\sigma_1\cdot(\sigma_1\sigma_2\sigma_1) = (\sigma_1\sigma_2\sigma_1)\cdot\sigma_2\sigma_1\sigma_2^3\sigma_1\sigma_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1\sigma_2^2\sigma_1^2 = \sigma_1\sigma_2\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1^2 = M_3(r),$$

where the second equality is conjugation by $\Delta = \sigma_1\sigma_2\sigma_1$ and the third and fourth are Reidemeister III moves. The Le diagram ⅃ associated to this positroid data in $\mathrm{Gr}(3,7)$ has partition $\lambda = (4,4,2)$ and is depicted in Figure 6. This figure also depicts the smooth braid associated to ⅃.

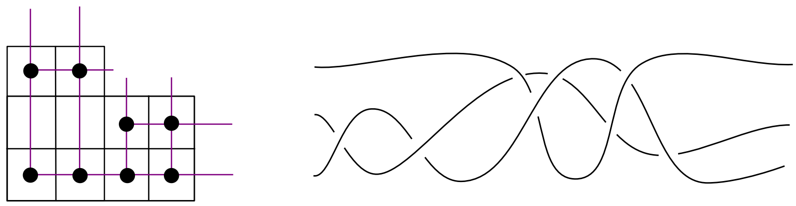


FIGURE 6. The Le diagram ⅃ associated to this introductory example with $(k,n) = (3,7)$ and partition $\lambda = (4,4,2)$, on the left. The 3-stranded Le braid word associated to this ⅃, on the right. See Section 3.4 for the general construction of a k-stranded braid word from a Le diagram inside the Young diagram $\lambda = (n-k)^k$.

Lastly, the Legendrian link $\Lambda(u,w) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, which is Legendrian isotopic to $\Lambda(f)$, $\Lambda(r)$ and $\Lambda(⅃)$, is the unique max-tb Legendrian representative of the link of the D_2-singularity $F(x,y) = (x^2+1)y$, or equivalently the $(A_1 \times A_1)$-singularity; see [10] for details on this Legendrian link. Notice that $\Lambda(u,w)$ admits four immediate embedded exact Lagrangian fillings L, smoothly isotopic and with the topology of a thrice-punctured sphere. These fillings yield four different cluster charts $(\mathbb{C}^*)^{b_1(L)} = (\mathbb{C}^*)^2$ in the augmentation variety $\mathrm{Aug}(\Lambda(u,w))$, and thus – upon stabilizing with frozens – in the corresponding positroid stratum $\Pi_{u,w} \subseteq \mathrm{Gr}(3,7)$. This matches the fact that the quiver associated to the Le diagram ⅃ in Figure 6, and the brick quiver of any of the braids, is of $A_1 \times A_1$ type, as is the mutable part of the quiver for the cluster structure in $\Pi_{u,w} \subseteq \mathrm{Gr}(3,7)$ given via plabic graphs. □

Organization: Section 2 constructs the pair $(X(\eta), V(\eta))$, studies its properties and proves Theorem 1.5. Theorem 1.3 is proven in Section 3, where the Le-braid is also introduced. Theorems 1.7 and

1.8 are proven in Section 4. Section 5 concludes this manuscript with a few conjectures on cluster $\mathcal{A}$-structures on braid varieties. □

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Notational Conventions: The grading of a DG-algebra will always be integral, i.e. $\mathbb{Z}$-grading, and a *graded commutative* DG-algebra will be referred to as simply a *commutative* DG-algebra. The differential decreases the grading by 1.

We use the notation $[i, j] := \{n \in \mathbb{N} : i \leq n \leq j\}$ for discrete intervals and denote $\sigma_{[a,b]} = \sigma_b \cdots \sigma_a$, for $a \leq b$. Given a braid word β, read left to right, its *opposite* $\reflectbox{$\beta$}$ is defined to be the braid word β read right to left, in reverse order. The half-twist word we use will be denoted by

$$\Delta_n := (\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \cdot \ldots \cdot (\sigma_{n-1}\sigma_{n-2} \ldots \sigma_2\sigma_1) \in \mathcal{B}_n^+,$$

and its Coxeter projection is denoted by $w_{0,n} \in S_n$. If $u, w \in S_n$, we denote $u \leq w$ if u is less than w in the Bruhat order. We will sometimes write $\beta_1 = \beta_2$ when two braid elements $\beta_1, \beta_2 \in \mathrm{Br}$ are conjugate, since we implicitly work with braid closures. Finally, given a matrix $A \in \mathrm{GL}(n, R)$, for a ground ring R, we denote by $A_{i,j}$ its (i, j) entry. □

2. Braid Varieties and DG-Algebras

This section constructs the braid pair $(X(\eta), V(\eta))$ associated to a braid word $\eta \in \mathcal{B}$ and proves Theorem 1.5. This is achieved by first defining a DG-algebra $\mathcal{A}(\eta\Delta)$ associated to η and then finding a geometric model that computes its zeroth cohomology group $H^0(\mathcal{A}(\eta\Delta))$. The invariance of the quotient $X(\eta)/V(\eta)$ associated to the braid pair, Theorem 1.5, is proven in Subsection 2.4, once all the necessary ingredients have been constructed. The symplectic topological model of this DG-algebra, based on the count of holomorphic strips, is proven in Theorem 2.16 in Subsection 2.5.

2.1. DG-algebra Ingredients.

Let us first introduce the notions and results on DG-algebras that we will need. The braid pair $(X(\eta), V(\eta))$ in the next subsection is constructed from an explicit and particular DG-algebra $\mathcal{A}$, and the main task will be computing the zeroth cohomology group $H^0(\mathcal{A})$. In general, it is challenging to compute the cohomology of a DG-algebra; fortunately, the DG-algebras that we associate to a braid word η, $[\eta] \in \mathrm{Br}^+$, are of the following particular sort:

Definition 2.1. A DG-algebra $(\mathcal{A}, \partial)$ is said to be *first order* if it is commutative, freely generated by generators of degrees $1, 0$ and -1, denoted $\{y_i\}_{i\in Y}$, $\{z_j\}_{j\in Z}$ and $\{w_k\}_{k\in W}$ respectively, $Y, Z, W \subseteq \mathbb{N}$, and the the differentials of the generators read

$$\partial(y_i) = f_i(z), \quad \partial(z_j) = \sum_{k\in W} g_{jk}(z)w_k, \qquad \partial(w_k) = 0, \qquad \text{for some } f_i, g_{jk} \in \mathbb{C}[z],$$

for all $(i, j, k) \in Y \times Z \times W$, and where z denotes the tuple $(z_j)_{j\in Z}$.

Furthermore, a DG-algebra $(\mathcal{A}, \partial)$ is $\mathbb{R}_{\geq 0}$-filtered if it has an algebra $\mathbb{R}_{\geq 0}$-filtration $h : \mathcal{A} \longrightarrow \mathbb{R}^+$ such that all generators are endowed with strictly positive filtration degree $h(y_i)$, $h(z_j)$ and $h(w_k)$ and the filtration satisfies $h(\partial(x)) \leq h(x)$ for all $x \in \mathcal{A}$. □

Should a commutative DG-algebra $(\mathcal{A}, \partial)$ be non-negatively graded and freely generated in degrees $0, 1$, the zeroth cohomology group $H^0(\mathcal{A})$ is generated by the degree 0 generators $\{z_j\}_{j\in Z}$ modulo the ideal $\langle \partial y_1, \ldots, \partial y_Y \rangle$ spanned by the differentials of the degree 1 generators $\{y_i\}_{i\in Y}$. Geometrically, the degree 0 generators can be considered as Cartesian coordinates in the affine space $\mathbb{C}^Z = \mathrm{Spec}(\mathbb{C}[z_1, \ldots, z_Z])$ and then $\mathrm{Spec}(H^0(\mathcal{A}))$ is the affine subvariety cut out by the differentials

$\{\partial y_i\}$, $i \in Y$. For any first order DG-algebra, as in Definition 2.1 above, we still consider the affine variety

$$\mathcal{H}^0(\mathcal{A}) := \mathrm{Spec}(\mathbb{C}[z_1, \ldots, z_Z]/\langle \partial y_1, \ldots, \partial y_Y \rangle),$$

but, in general, $\mathcal{H}^0(\mathcal{A}) \neq \mathrm{Spec}(H^0(\mathcal{A}))$. The generators w_k correspond to vector fields

$$V_k = \sum g_{jk}(z) \frac{\partial}{\partial z_j}$$

and it is easy to see that the equation $\partial^2 = 0$ implies that these vector fields commute and preserve the variety $\mathcal{H}^0(\mathcal{A})$. Thus, we encode the data of a DG-algebra $(\mathcal{A}, \partial)$ as the affine subvariety $\mathcal{H}^0(\mathcal{A}) \subseteq \mathbb{C}^Z$ *endowed* with a set of commuting tangent vector fields.

We will also use the following algebraic facts. Let A be a commutative algebra, then derivations on A correspond to vector fields on $\mathrm{Spec}\, A$ and the kernel of a derivation V is defined as

$$\mathrm{Ker}(V) = \{a : V(a) = 0\}.$$

It is easy to see that $\mathrm{Ker}(V)$ is a subalgebra of A. By definition, a derivation V is said to be *locally nilpotent* if for all $a \in A$ one has $V^n(a) = 0$ for sufficiently large n (depending on a). If the algebra A is commutative, then V is locally nilpotent if and only if the corresponding vector field on $\mathrm{Spec}\, A$ integrates to an *algebraic* action of $\mathbb{C}$. For instance, ∂_x is a locally nilpotent derivation on $\mathbb{C}[x]$ while $x\partial_x$ is not locally nilpotent; correspondingly, the vector field ∂_x yields the algebraic action $t.x = x + t$ on $\mathrm{Spec}\, \mathbb{C}[x] = \mathbb{C}$, whereas $x\partial_x$ corresponds to the non-algebraic action $t.x = e^t x$.

Given a derivation V on an algebra A, we call an element $x \in A$ a *slice* for V if $V(x) = 1$. The following fact, also known as Slice Theorem [31, Corollary 1.26], will be useful for us:

Proposition 2.2. *Let A be an algebra, $V : A \longrightarrow A$ a locally nilpotent derivation, and $x \in A$ a central element that is also a slice for V. Then the following holds:*

(i) We have an algebra isomorphism $A \simeq \mathbb{C}[x] \otimes \mathrm{Ker}(V)$.

(ii) If the algebra A is commutative, then the action of V is free and we have an isomorphism

$$\mathrm{Spec}\, A \cong \mathrm{Spec}(\mathrm{Ker}(V)) \times \mathbb{C}.$$

Proof. For Part (i), the action of V and (left) multiplication by x satisfy the Heisenberg algebra relation $[V, x] = 1$. Indeed,

$$V(xa) = V(x)a + xV(a) = a + xV(a).$$

If V is locally nilpotent then the corresponding module over the Heisenberg algebra is free over x, and the natural map $\mathbb{C}[x] \otimes \mathrm{Ker}(V) \to A$ is an isomorphism of vector spaces. Since x is central in A, this is an algebra isomorphism. The statement in Part (ii) readily follows from Part (i). □

The following proposition shows that, assuming that the first order DG-algebra $\mathcal{A}$ is filtered and has no negatively graded cohomology, $\mathrm{Spec}(H^0(\mathcal{A}))$ can still be geometrically described in terms of the affine subvariety $\mathcal{H}^0(\mathcal{A}) \subseteq \mathbb{C}^Z$ and tangent vector fields.

Proposition 2.3. *Let $(\mathcal{A}, \partial)$ be a first order filtered DG-algebra with $H^{-1}(\mathcal{A}) = 0$. Then there exists a free $\mathbb{C}^W$-action on the affine subvariety $\mathcal{H}^0(\mathcal{A}) \subseteq \mathbb{C}^Z$ such that*

$$\mathrm{Spec}(H^0(\mathcal{A})) \cong \mathcal{H}^0(\mathcal{A})/\mathbb{C}^W$$

as affine varieties. In fact, the $\mathbb{C}^W$-action is generated by a set of globally commuting vector fields defined in affine space $\mathbb{C}^Z$ tangent to the affine subvariety $\mathcal{H}^0(\mathcal{A}) \subseteq \mathbb{C}^Z$.

Proof. Let us show that, with the hypothesis of $H^{-1}(\mathcal{A}) = 0$, there exists a non-negatively graded algebra $\mathcal{A}'$, elements $\alpha_1, \ldots, \alpha_W \in \mathcal{A}$ such that $\partial \alpha_k = w_k$, $1 \leq k \leq |W|$, and an algebra isomorphism

$$\mathcal{A} \cong \mathcal{A}' \otimes \mathbb{C}[\alpha_1, \ldots, \alpha_W, w_1, \ldots, w_W],$$

where the differential sends $\partial(\mathcal{A}') \subseteq \mathcal{A}' \otimes \mathbb{C}[\alpha_1, \ldots, \alpha_W]$. This implies the desired statement, since $\mathrm{Ker}(\partial|_{\mathcal{A}_0}) = \mathcal{A}'_0$ and thus $H^0(\mathcal{A}) \cong H^0(\mathcal{A}')$, where $\mathcal{A}'_0$ is the degree 0 part of the graded algebra $\mathcal{A}'$. In order to construct the above isomorphism, we first notice that the elements $\alpha_1, \ldots, \alpha_W$ exist because $H^{-1}(\mathcal{A}) = 0$ vanishes and the degree -1 generators $w_1, \ldots, w_W$ are closed. Thus, we must argue that $\mathcal{A}$ is free over the polynomial ring $\mathbb{C}[\alpha_1, \ldots, \alpha_W, w_1, \ldots, w_W]$. This can be done iteratively, peeling off one $\mathbb{C}[\alpha_k, w_k]$ factor at a time, as follows.

By hypothesis, the algebra $\mathcal{A}$ is free over the element $w := w_W$. This element defines a derivation ∂_w, by declaring $\partial_w(x)$ to be the coefficient in front of w of the differential $\partial(x)$. Since $\mathcal{A}$ is a filtered algebra, the derivation is locally nilpotent and, by construction, $\alpha := \alpha_W$ is a slice for the $\mathbb{C}$-action that ∂_w induces. Since α has degree 0, it is central, and thus Proposition 2.2. (i) implies that the quotient $\mathcal{A}/\langle w\rangle \cong \ker(\partial_w) \otimes \mathbb{C}[\alpha]$ is isomorphic to the tensor product of the kernel $\ker(\partial_w)$ of the locally nilpotent derivation ∂_w and the polynomial ring $\mathbb{C}[\alpha]$. This implies $\mathcal{A} \cong \mathcal{A}' \otimes \mathbb{C}[\alpha, w]$, where $\mathcal{A}' := \ker(\partial_w)$. It is readily verified that $w_1, \ldots, w_{W-1}, \alpha_1, \ldots, \alpha_{W-1} \in \mathcal{A}'$. This allows us to iterate this process until we achieve the required isomorphism. □

2.2. **Braid Ingredients.** Let us continue introducing the necessary concepts in relation to braids.

Consider a braid word

$$\beta = \sigma_{i_1}^{\varepsilon_{i_1}} \sigma_{i_2}^{\varepsilon_{i_2}} \cdot \ldots \cdot \sigma_{i_\ell}^{\varepsilon_{i_\ell}}, \qquad \varepsilon_{i_p} \in \{\pm 1\}, \quad 1 \leq i_p \leq n, \quad 1 \leq p \leq l,$$

and associate the variable z_j to the jth positive crossing of β, and the variable w_k to the kth negative crossing of β, read left to right. By definition, the braid word $\beta(z_j, w_k)$ is the braid subword of β strictly between the jth positive crossing and the kth negative crossing. Similarly, $\beta^c(z_j, w_k)$ denotes the braid word obtained by removing the braid subwords $\beta(z_j, w_k), z_j, w_k$ from β and reading left to right the resulting braid word, by starting at the first crossing to the right of whichever crossing z_j or w_k is rightmost, then cyclically continuing to the beginning when the rightmost crossing of β is reached, and continuing until the first crossing to the left of the *other* crossing (the leftmost one between z_j and w_k) is found.

Denote by $\dot{\beta}(z_j, w_k)$ the *opposite* of the braid word $\beta(z_j, w_k)$ where each crossing $\sigma_{i_p}^{\varepsilon_{i_p}}$ is changed to $\sigma_{n-i_p}^{\varepsilon_{i_p}}$; the opposite braid word is just $\beta(z_j, w_k)$ read right to left. Also, denote by $\beta_L(w_k)$, respectively $\beta_R(w_k)$ the braid subword of β strictly to the left of w_k, respectively strictly to the right of w_k.

Example 2.4. *Consider the braid word*

$$\beta = \sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_3\sigma_1 \in \mathcal{B}_3,$$

which has associated the variables $z_1, w_1, z_2, z_3, z_4, z_5, w_2, z_6, w_3, z_7, z_8, w_4, z_9, z_{10}$, reading the crossings left to right. Same instances of braid subwords are

$$\beta(z_2, w_3) = \sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_3, \quad \beta(z_7, w_1) = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_1^{-1}.$$

Their inverted opposites read

$$\dot{\beta}(z_2, w_3) = \sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_2, \quad \dot{\beta}(z_7, w_1) = \sigma_3^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_2\sigma_1,$$

and their complements are

$$\beta^c(z_2, w_3) = (\sigma_3\sigma_2\sigma_1^{-1}\sigma_3\sigma_1)(\sigma_1\sigma_2^{-1}), \quad \beta^c(z_7, w_1) = (\sigma_2\sigma_1^{-1}\sigma_3\sigma_1)(\sigma_1\sigma_2^{-1}).$$

We also have the braids

$$\beta_L(w_2) = \sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1\sigma_2, \quad \beta_R(w_2) = \sigma_3\sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_3\sigma_1.$$

These types of braid words associated to a braid β feature in the definition of the DG-algebra in Subsection 2.3. □

Consider the matrix $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ defined by:

$$(B_i(z))_{jk} := \begin{cases} 1 & j = k \text{ and } j \neq i, i+1 \\ 1 & (j,k) = (i, i+1) \text{ or } (i+1, i) \\ z & j = k = i+1 \\ 0 & \text{otherwise;} \end{cases}, \quad \text{i.e. } B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

This is the *braid matrix* in the introduction. Let us denote

$$\chi(\sigma_j, \varepsilon) := \begin{cases} z_j & \text{if } \varepsilon = 1 \\ 0 & \text{if } \varepsilon = -1 \end{cases}.$$

Definition 2.5. Let $\beta \in \mathcal{B}_n$ be a braid word
$$\beta = \sigma_{i_1}^{\varepsilon_{i_1}} \sigma_{i_2}^{\varepsilon_{i_2}} \cdot \ldots \cdot \sigma_{i_\ell}^{\varepsilon_{i_\ell}}, \qquad \varepsilon_{i_p} \in \{\pm 1\}, \quad 1 \leq i_p \leq n, \quad 1 \leq p \leq l.$$
The braid matrix $B_\beta(z_1, \ldots, z_Z)$ is defined to be the product
$$B_\beta(z_1, \ldots, z_Z) := \prod_{1 \leq p \leq \ell} B_{i_p}(\chi(\sigma_{i_p}, \varepsilon_{i_p})).$$
In words, the braid matrix $B_\beta(z_1, \ldots, z_Z)$ associated to β is obtained by multiplying the braid matrices for each of its crossings, left to right, where the jth positive crossing σ_i contributes with a braid matrix $B_i(z_j)$ and the kth negative crossing σ_l^{-1} contributes with a braid matrix $B_l(0)$. □

The additive terms in the (i,j)-entry $B_\beta(z_1, \ldots, z_Z)_{i,j}$ of the braid matrix $B_\beta(z_1, \ldots, z_Z)$ correspond bijectively to *paths* from the leftmost end of the ith strand to the rightmost end of the jth strand. A *path* γ is defined to be a path contained in the braid diagram, moving left to right, which abides by the following local rule: if γ is at the $(i+1)$th strand and encounters a *positive* σ_i-crossing, then it either remains in the same $(i+1)$th strand moving past the crossing (i.e. it jumps above the crossing), or moves downwards (through the crossing) to the ith strand. In any other situation, including encountering a negative crossing, the path must just continue through the crossing, and therefore switch level, to either the $(i+2)$th strand, if $\sigma_{i+1}^{\pm}$ is encountered or the ith strand, if σ_i^{-1} is encountered. The bijection between each monomial term in the (i,j)-entry $B_\beta(z_1, \ldots, z_Z)_{i,j}$ and paths γ from the ith strand on the left to the jth strand on the right is given by recording the crossings where the path jumps, staying at the same level strand: the product of these crossings yields a monomial term in $B_\beta(z_1, \ldots, z_Z)_{i,j}$ and, by construction, this is a bijection.

The geometric path associated to monomial $\mathfrak{m}$ in $B_\beta(z_1, \ldots, z_Z)_{i,j}$ will be denoted by $\gamma(\mathfrak{m})$. Note that $\gamma(1)$ is a well-defined path, where no jumps are taken: the path starts tracing the ith strand from the left and always remains in that connected component, moving through all the crossings it encounters. Finally, consider two paths γ_1, γ_2 on a braid word β, where γ_1 starts at the ith strand on the left and finishes at the jth strand on the right, and γ_2 starts at the $(i+1)$th strand on the left and finishes at the $(j+1)$th strand on the right. By definition, the pair (γ_1, γ_2) is said to define an *unoriented immersed region* $\mathcal{R}(\gamma_1, \gamma_2)$ if the paths γ_1 and γ_2 bound the projection of an immersed 2-dimensional disk into the plane $\mathbb{R}^2$ with boundary on the braid diagram.

Definition 2.6. An unoriented immersed region $\mathcal{R}(\gamma_1, \gamma_2)$ is said to be orientable if its defining paths (γ_1, γ_2) can be oriented such that the boundaries of the region $\mathcal{R}(\gamma_1, \gamma_2)$ are oriented according to the local models in Figure 7. Note that these local models only apply if the region is (locally) to the left, or right, of a positive or negative crossings.

By definition, an immersed region is an unoriented immersed region which is orientable. □

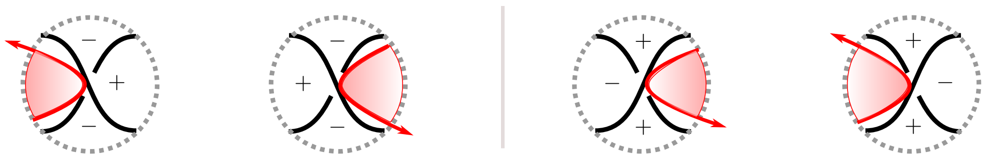


FIGURE 7. The allowed local orientations for an *oriented* immersed region.

Let us now define a certain quantity that records these (oriented) immersed regions bounded by two paths with fixed endpoints. Let $i_1^u, i_1^l, i_2^u, i_2^l \in [1, n]$ index strands and $\eta \in \mathcal{B}_n$ be a positive braid word; intuitively, i_1^l, i_1^u are indexing two strands to the left of the regions, *lower* and *upper*, and i_2^l, i_2^u are indexing two strands to the right of the regions. The coefficient $\mathcal{E}(B(\eta); i_1^u, i_1^l, i_2^u, i_2^l)$ is described as follows. Consider the product
$$c(\eta; i_1^u, i_1^l, i_2^u, i_2^l) := (B_\eta)_{i_1^l, i_2^l} \cdot (B_{\dot{\eta}})_{n-i_1^u+1, n-i_2^u+1}$$
of the (i_1^l, i_2^l)-entry of the braid matrix associated to the braid word η and the $(n-i_1^u+1, n-i_2^u+1)$-entry of the braid matrix associated to the braid word $\dot{\eta}$. By definition, the coefficient $\mathcal{E}(B(\eta); i_1^u, i_1^l, i_2^u, i_2^l)$ is the sum of the terms in $c(\eta; i_1^u, i_1^l, i_2^u, i_2^l)$ which are the product of two monomials $\mathfrak{m}_1, \mathfrak{m}_2$, where $\mathfrak{m}_1$

is a monomial in $(B_\eta)_{i_1^l,i_2^l}$ and $\mathfrak{m}_2$ a monomial in $(B_{\dot\eta})_{n-i_1^u+1,n-i_2^u+1}$ such that the pair of associated paths $(\gamma(\mathfrak{m}_1),\gamma(\mathfrak{m}_2))$ defines an immersed region in η. We will abbreviate $\mathcal{E}(B(\eta);i_1^u,i_1^l,i_2^u,i_2^l)$ to $\mathcal{E}(B(\eta);i_1,i_2)$, if $i_1=i_1^l=i_1^u-1$ and $i_2=i_2^l=i_2^u-1$. These quantities appear in the differential of the DG-algebra in Subsection 2.3.

Lastly, a braid word $\beta\in\mathcal{B}_n$ was said to be *admissible* in [12, Definition 2.5] if there existed braid words $\eta_1,\eta_2\in\mathcal{B}_n$ such that $\beta=\eta_1\Delta_n\eta_2$, i.e. if β contains Δ_n as a subword. We will always use admissible words $\beta\in\mathcal{B}_n$ with trivial η_2, and thus, just for this article, we say that a braid word $\beta\in\mathcal{B}_n$ is *admissible* if it is of the form $\beta=\eta\Delta_n\in\mathcal{B}_n$ for some $\eta\in\mathcal{B}_n$ and $[\eta]\in[\mathrm{Br}_n^+]$.

2.3. **The braid pair** $(X(\eta),V(\eta))$**.** In this section we first associate a DG-algebra $\mathcal{A}(\beta)$ to admissible braid words $\beta=\eta\Delta_n\in\mathcal{B}_n$. Then, we introduce the braid pair $(X(\eta),V(\eta))$, which is defined using $\mathcal{A}(\eta\Delta_n)$. Let us introduce the DG-algebra by describing its generators, their grading and the differential.

Generators and Degrees. By definition, the DG-algebra $\mathcal{A}(\beta)$ is the graded commutative algebra over the ground ring $R=\mathbb{Z}[\mathbb{Z}^n]$ generated by n^2 generators y_{lm} in degree 1, $1\le l,m\le n$, $|Z|$ generators $z_1,\dots,z_Z$ in degree 0 and $|W|$ generators $w_1,\dots,w_W$ in degree -1, where $|Z|$ denotes the number of positive crossings in β and $|W|$ denotes the number of negative crossings. Thus, as a graded algebra, $\mathcal{A}(\beta)$ is rather simple: all the intricate information is contained in the graded differential ∂. Let us fix an isomorphism $R\cong\mathbb{Z}[t_1^\pm,\dots,t_n^\pm]$ of the ground ring with the Laurent polynomial ring in the ground variables $t_1,\dots,t_n$.[4]. □

The Differential. The differential can be extended to $\mathcal{A}(\beta)$ by the Leibniz rule once it has been defined on the generators. The differential of the degree 1 generators y_{lm} is given by

$$\partial y_{lm}:=B_\beta(z_1,\dots,z_Z)_{l,m}+\delta_{l,m}t_l+Ш(y_{lm}),\quad 1\le l,m\le n,$$

where the braid matrix $B_\beta(z_1,\dots,z_Z)_{l,m}$ has been introduced in Definition 2.5 above and $\delta_{l,m}$ is the Kronecker delta. The term $Ш(y_{lm})$ will be described momentarily. Nevertheless, see Remarks 2.7 and 2.8 below noting that the Ш-term directly vanishes in many interesting cases and can otherwise be removed by using an automorphism of the DG-algebra; we still define it for completeness.

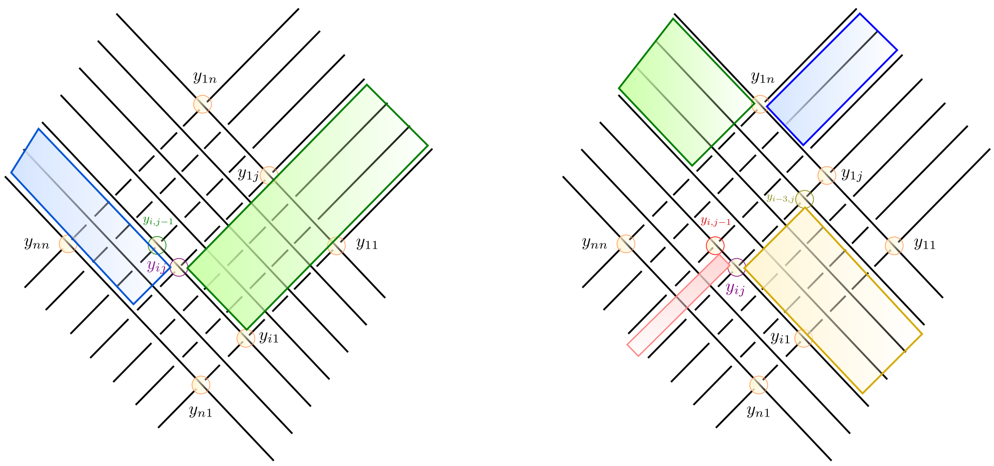


FIGURE 8. A pictorial depiction of the generators y_{lm}, $l,m\in[1,n]$ and their differentials. On the left, regions contributing to the $Ш(y_{ij})$ term in the differential $\partial(y_{ij})$, drawn in green and blue. On the right, regions contributing to the $Ш(y_{1n})$ term in the differential $\partial(y_{1n})$, drawn in green and blue. In contrast, the differential $\partial_{y_{ij}}$ does *not* include the contributions from the red and yellow regions drawn on the right, due to the admissibility hypothesis on β.

[4]In many cases, it is equally useful to consider the DG-algebra over $\mathbb{Z}$, instead of R. Should the reader feel more comfortable this way, they may set the ground variables $t_1,\dots,t_n$ to ±1 such that the product of all the t-variables in the same connected component multiply to -1.

For each $l,m \in [1,n]$, and degree -1 generator w_k we write

$$Ш(y_{lm};w_k) := Ш_L(y_{lm};w_k) + Ш_R(y_{lm};w_k),$$

where

$$Ш_L(y_{lm};w_k) = \sum_{l<p\leq n} \mathcal{E}(B(\beta_L(w_k)); l,p,\iota(w_k),\iota(w_k)+1)y_{pm},$$

$$Ш_R(y_{lm};w_k) = \sum_{1\leq p<m} \mathcal{E}(B(\beta_L(w_k)); p,m,\iota(w_k),\iota(w_k)+1)y_{lp}.$$

Then, the coefficient $Ш(y_{lm})$ is simply given by

$$Ш(y_{lm}) := \sum_{1\leq k\leq W} Ш(y_{lm};w_k)w_k.$$

The term $Ш(y_{lm};w_k)$ is computing regions that start at y_{lm}, as depicted in Figure 8, continue to the left of the braid β, and end at the negative crossing w_k. This gives a simple combinatorial intuition behind the rather algebraic definition of $Ш(y_{lm})$. Only half of the regions starting at y_{lm} are accounted for due to the presence of Δ_n to the right of the admissible word β; we could have expanded $Ш(y_{lm})$ to account for all the regions starting at y_{lm}, but the corresponding coefficients $\mathcal{E}(B(\beta_R(w_k)); l,p,\iota(w_k),\iota(w_k)+1)$ and $\mathcal{E}(B(\beta_R(w_k)); p,m,\iota(w_k),\iota(w_k)+1)$ would have been all zero. This concludes the description of the differential of the degree 1 generators.

The differential of the degree 0 generators z_j, $1\leq j\leq |Z|$, is given by

$$\partial z_j = \sum_{1\leq k\leq |W|} \mathrm{Sign}(z_j,w_k)\left[\mathcal{E}(B(\beta(z_j,w_k));\iota(z_k),\iota(w_k)) - \mathcal{E}(B(\beta^c(z_j,w_k));\iota(z_k),\iota(w_k))\right]\cdot w_k,$$

where $\mathrm{Sign}(z_j,w_k)\in\{\pm 1\}$ is one if z_j is to the left of w_j and minus one otherwise. The coefficients $\mathcal{E}(B(\beta(z_j,w_k));\iota(z_k),\iota(w_k))$ and $\mathcal{E}(B(\beta^c(z_j,w_k));\iota(z_k),\iota(w_k))$ have been defined in Subsection 2.2 and we have denoted $\iota(a)=i$ if the a-variable is assigned to a σ_i or σ_i^{-1} crossing.

The differential of the degree -1 generators w_k, $1\leq k\leq |W|$, is defined to be

$$\partial w_1 = 0, \quad \partial w_2 = 0, \quad \ldots, \quad \partial w_W = 0.$$

That is, *all* degree -1 generators have zero differential. □

Remark 2.7. The terms $Ш(y_{lm})$ in the differentials ∂y_{lm} vanish for many interesting cases. For instance, if $\beta\in\mathcal{B}_n$ is of the form $\beta=\Delta_n\eta\Delta_n$, for some $\eta\in\mathcal{B}_n$, then $Ш(y_{lm})=0$ for all indices l,m in their domain. Note that the Richardson braid $R_n(u,w)\Delta_n^2$, that gives the Legendrian link $\Lambda(u,w)$, yields the braid word $\Delta_n R_n(u,w)\Delta_n$ in Theorem 1.3, after conjugation. Hence, all computations related to open positroid varieties can be performed with the braid word $\Delta_n R_n(u,w)\Delta_n$ and the Ш-terms vanish. □

This defines the braid DG-algebra $\mathcal{A}(\beta)$ for any admissible braid word $\beta\in\mathcal{B}_n$ such that $[\beta]\in\mathrm{Br}_n^+$. The fact that $\partial^2=0$ can be deduced from the combinatorial argument in [15]; this also implies that $\mathcal{A}(\beta)$ can be $\mathbb{R}_{\geq 0}$-filtered by the integral of the Liouville form at each crossing. For these DG-algebras, the $\mathbb{R}_{\geq 0}$-filtration $h:\mathcal{A}\longrightarrow\mathbb{R}_{\geq 0}$ can be taken to be any positive function such that

$$0<h(w_k)<h(z_j)<h(y_{lm}),\ \forall l,m,j,k, \quad \text{and} \quad h(y_{pq})<h(y_{lm}),\ \text{if } q<p+(m-l).$$

Any such filtration will be fixed and implicitly chosen throughout the computations.

Remark 2.8. Note that the DG-algebra $\mathcal{A}(\beta)$, as presented, is not of first order, as in Definition 2.1, due to the appearance of the terms $Ш(y_{lm})$ in the differentials of the degree 1 generators. That said, the differential ∂y_{lm} only contains terms of the form $y_{pq}w_k$, which are linear in w_k and satisfy $h(y_{pq})<h(y_{lm})$, by construction. Thus, if $H^{-1}(\mathcal{A}(\beta))=0$ vanishes, there exists an automorphism of $\mathcal{A}(\beta)$ fixing the degree 0 and degree -1 generators, and only modifying the degree 1 generators $y_{lm}\longmapsto\tilde{y}_{lm}$, such that the new degree 1 generators $\tilde{y}_{lm}$ have no degree -1 term in their differentials $\partial(\tilde{y}_{lm})$, i.e. $\partial(\tilde{y}_{lm})\in\mathbb{C}[z_1,\ldots,z_Z]$. This can be proved similarly to Proposition 2.3. Hence we will implicitly assume that this change of coordinates has been performed and $\mathcal{A}(\beta)$ is of first order. Note that for braid words $\beta=\Delta_n\eta\Delta_n$ which also have a Δ_n to their left, this automorphism is the identity. □

Two simple examples are presented in Examples 2.10 and 2.12. The DG-algebra associated to *any* 2-stranded word $\beta \in \mathcal{B}_2$ is computed by readily generalizing Example 2.12. The main algebraic structure that we use for this DG-algebra is its zeroth degree cohomology $H^0(\mathcal{A}(\beta))$, which we now discuss.

Following Subsection 2.1 above, we consider the affine variety $\mathcal{H}^0(\mathcal{A})$ associated to a commutative DG-algebra, which in the context of a braid DG-algebra $\mathcal{A}(\beta)$, for $\beta = \eta\Delta_n$, we denote $X(\eta)$. It is an affine R-scheme given by the equations:

$$X(\eta) := \{(z_1, \ldots, z_Z) \in \mathbb{C}^Z : B_\beta(z_1, \ldots, z_Z)_{l,m} + \delta_{l,m} t_l = 0, \quad 1 \le l, m \le n\} \subseteq \mathbb{C}^Z.$$

It is, in general, not true that $X(\eta) \cong \operatorname{Spec} H^0(\mathcal{A}(\beta))$, e.g. see Example 2.12 below. Nevertheless, $\mathcal{A}(\beta)$ is an algebra of first order, as in Definition 2.1, and thus $\operatorname{Spec} H^0(\mathcal{A}(\beta))$ can be computed by incorporating a set of commuting vector fields $V(w_1), \ldots, W(w_W)$ associated to the degree -1 generators. Indeed, Proposition 2.3 implies that

$$\operatorname{Spec} H^0(\mathcal{A}(\beta)) = \operatorname{Spec} \left(\operatorname{Ker}\{V(w_1), \ldots, V(w_W)\}\right).$$

This leads us to the following definition, using the same notation as above:

Definition 2.9. Let $\eta \in \mathcal{B}_n$ be a braid word such that $[\eta] \in \mathrm{Br}_n^+$ and let $\beta := \eta\Delta_n$. By definition, the *braid pair* $(X(\eta), V(\eta))$ consists of the affine variety

$$X(\eta) := \{(z_1, \ldots, z_Z) \in \mathbb{C}^Z : B_\beta(z_1, \ldots, z_Z)_{l,m} + \delta_{l,m} t_l = 0, \quad 1 \le l, m \le n\} \subseteq \mathbb{C}^Z,$$

endowed with the set $V(\eta) := \{V(w_1), \ldots, V(w_W)\}$ of tangent vector fields defined by

$$V(w_k) := \sum_{1 \le j \le |Z|} \operatorname{Sign}(z_j, w_k) \mathcal{E}(B(\beta(z_j, w_k))) \cdot \partial_{z_j}, \quad 1 \le k \le |W|.$$

By definition, the *braid variety* associated to $\eta \in \mathcal{B}_n$ is the affine quotient $X(\eta)/V(\eta)$ of $X(\eta)$ by the $\mathbb{C}^W$-action induced by the commuting vector fields in $V(\eta)$. □

Let us conclude this subsection with two simple examples, which hopefully illustrate the concepts and definitions introduced thus far.

Example 2.10. *Consider the braid* $\beta = \sigma_1\sigma_3\sigma_2^3\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_3^2\sigma_1^{-1}$ *depicted in Figure 9. The leftmost crossing carries the variable* z_1 *and the rightmost crossing the variable* w_1*. In this case* $\beta(z_1, w_1) = \sigma_3\sigma_2^3\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_3^2$ *. The crossing* z_1 *is between strands* $1, 2$*, i.e. of* σ_1*-type, and is a generator of degree 0. Its differential includes a term of the form* $\partial(z_1) = c \cdot w_1 + \ldots$*, where* w_1 *is a (negative) crossing between strands* $1, 2$*, i.e. of* σ_1^{-1}*-type and a generator of degree* -1*. The coefficient* c *in the differential of* z_1 *is computed by noticing that* $B(\beta(z_1, w_1))_{1,1} = 1$ *and that the only terms in* $B(\dot\beta(z_1, w_1))_{3,3}$ *which yield immersed regions with respect to the unique term* $B(\beta(z_1, w_1))_{1,1} = 1$ *are* 1 *and* z_2z_3*. Hence we obtain* $\mathcal{E}(B(\beta(z_1, w_1))) = 1 + z_2z_3$ *and the* w_1*-coefficient of the differential is given by* $c = 1 + z_2z_3$*, i.e.*

$$\partial(z_1) = (1 + z_2z_3) \cdot w_1 + \ldots$$

The region associated to the second term z_2z_3 *in* $1 + z_2z_3$ *is depicted in red in Figure 9.* □

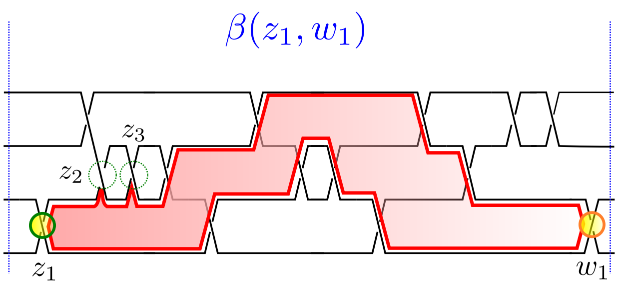


FIGURE 9. A region contributing to the differential of the degree 0 generator z_1 discussed in Example 2.10. The red region highlighted in the picture contributes to a summand $(z_2z_3)w_1$ in ∂z_1.

Example 2.11. *As a variation, consider the braid* $\beta = \sigma_2^{-1}\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2^{-1}\sigma_2\sigma_1\sigma_3\sigma_2$ *depicted in Figure 10. Label the crossings with* z_j*- and* w_k*-variables accordingly, the former for positive crossings and the latter for negative crossings, indices increasing left to right. Then*

$$\beta(z_9, w_1) = \sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2^{-1}\sigma_2\sigma_1\sigma_3.$$

The crossing z_9 *is between strands* $2, 3$*, i.e. of* σ_2*-type and a generator degree 0; its differential includes a term of the form* $\partial(z_9) = c \cdot w_1 + \ldots$*, where* w_1 *is a (negative) crossing between strands* $2, 3$*, i.e. of* σ_2^{-1}*-type and a generator of degree* -1*. In this case, we obtain*

$$B(\beta(z_9, w_1))_{2,2} = z_1 + z_7 + z_1 z_5 z_7$$

and $B(\dot\beta(z_9, w_1))_{2,2}$ *is similarly computed. The product of* z_2 *and* z_7 *is assigned to the embedded region depicted in Figure 10, and thus contributes to a summand* $-(z_2 z_7)w_1$ *in the differential* ∂z_9*.* □

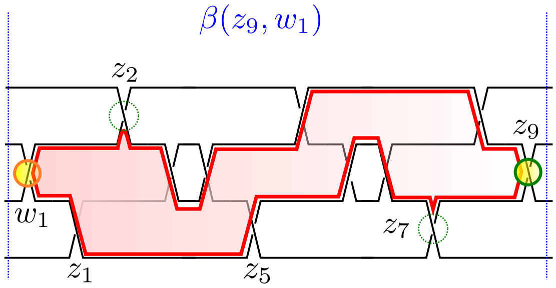


FIGURE 10. A region contributing to the differential of the degree 0 generator z_9, as discussed in Example 2.11. The red region contributes with $(z_2 z_7)w_1$ in ∂z_9.

Example 2.12. *Let us consider the admissible braid word* $\beta = \sigma_1\sigma_1^{-1}\sigma_1^5 = (\sigma_1\sigma_1^{-1}\sigma_1^4)\Delta_2 = \eta\Delta_2 \in \mathcal{B}_2$*,* $\eta = (\sigma_1\sigma^{-1}\sigma_1^4)$*, depicted in Figure 11, with* $[\beta] \in \mathrm{Br}_2^+$ *representing the trefoil knot with an inserted negative crossing, coming from a Reidemeister II move. For its associated DG-algebra* $\mathcal{A}(\beta)$*, there are four degree 1 generators* $y_{11}, y_{12}, y_{21}, y_{22}$*, six degree 0 generators* $z_1, z_2, z_3, z_4, z_5, z_6$ *and one generator* w_1 *of degree* -1*. For simplicity, we set the ground variables* $t_1 = 1, t_2 = 1$ *to one in this example. The differentials of the degree 1 generators can be computed directly from the four entries of the matrix product*

$$B_1(z_1)B_1(0)B_1(z_2)B_1(z_3)B_1(z_4)B_1(z_5)B_1(z_6)$$

and explicitly read

$$\partial y_{11} = z_3 + (1 + z_3 z_4)z_5, \quad \partial y_{12} = 1 + z_3 z_4 + (z_3 + (1 + z_3 z_4)z_5)z_6 + 1,$$
$$\partial y_{21} = 1 + (z_1 + z_2)z_3 + (z_1 + z_2 + (1 + (z_1 + z_2)z_3)z_4)z_5$$
$$\partial y_{22} = z_1 + z_2 + (1 + (z_1 + z_2)z_3)z_4 + (1 + (z_1 + z_2)z_3 + (z_1 + z_2 + (1 + (z_1 + z_2)z_3)z_4)z_5)z_6 + 1.$$

The differentials of the degree 0 generators are

$$\partial z_1 = w_1, \quad \partial z_2 = -w_1, \quad \partial z_i = 0, \quad 3 \le i \le 6.$$

The regions associated to the terms contributing to the two non-zero differentials ∂z_1 *and* ∂z_2 *are drawn in red in Figure 11. Note that in this case the intermediate braids* $\beta(z_1, w_1)$ *and* $\beta(z_2, w_1)$ *are both empty and all the associated braid matrices are the identity. Note that* $\partial^2 y_{11} = 0$ *and* $\partial^2 y_{12} = 0$ *are immediate, and* $\partial^2 y_{22} = 0$ *and* $\partial^2 y_{21} = 0$ *hold because, in addition,* $\partial(z_1 + z_2) = 0$*.*

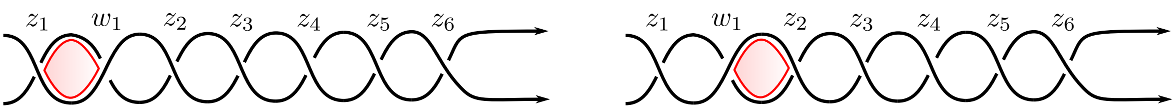


FIGURE 11. Embedded regions contributing to the differentials ∂z_1 and ∂z_2 for the braid word $\beta = \sigma_1\sigma_1^{-1}\sigma_1^5 \in \mathcal{B}_2$ discussed in Example 2.12. The red region on the left yields the term w_1 in ∂z_1, whereas the red region on the right gives w_1 in ∂z_2.

The affine variety $X(\eta) = \mathcal{H}^0(\mathcal{A})$ is cut out by the four equations

$$\tilde{X} := \{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 : \partial y_{11} = 0, \partial y_{12} = 0, \partial y_{21} = 0, \partial y_{22} = 0\} \subseteq \mathbb{C}^6,$$

coming from the degree 1 differentials, which can be explicitly read right above. The vector field associated to the negative crossing w_1 is given by $V(w_1) = \partial_{z_1} - \partial_{z_2} \in H^0(\mathbb{C}^6, T\mathbb{C}^6)$. Since $\partial y_{11}, \partial y_{12}$ contain no terms in z_1, z_2 and $\partial y_{21}, \partial y_{22}$ are functions on the sum $z_1 + z_2$ (and not just z_1, z_2), the Lie derivatives

$$\mathcal{L}_{V(w_1)}(\partial y_{ij}) = 0, \qquad 1 \leq i, j \leq 2,$$

vanish. Hence, the vector field $V(w_1) \in H^0(\tilde{X}, T\tilde{X})$ is tangent to $\tilde{X}$. This vector field integrates to the free $\mathbb{C}$-action given by

$$\mathbb{C} \times \tilde{X} \longrightarrow \tilde{X}, \quad (t, (z_1, z_2, z_3, z_4, z_5, z_6)) \longmapsto (z_1 + t, z_2 - t, z_3, z_4, z_5, z_6),$$

and the quotient $\tilde{X}/V(w_1)$ yields the affine variety

$$\tilde{X}/V(w_1) = \{(\zeta, z_3, z_4, z_5, z_6) \in \mathbb{C}^5 : \partial y_{11} = 0, \partial y_{12} = 0, f_1(\zeta, z_3, z_4, z_5, z_6) = 0, f_2(\zeta, z_3, z_4, z_5, z_6) = 0\}$$

in affine space $\mathbb{C}^5$, where $\partial y_{11}, \partial y_{11}$ are seen as functions on (z_3, z_4, z_5, z_6), and we have denoted

$$f_1(\zeta, z_3, z_4, z_5, z_6) := 1 + \zeta z_3 + (\zeta + (1 + \zeta z_3)z_4)z_5,$$

$$f_2(\zeta, z_3, z_4, z_5, z_6) := \zeta + (1 + \zeta z_3)z_4 + (1 + \zeta z_3 + (\zeta + (1 + \zeta z_3)z_4)z_5)z_6.$$

Note that this quotient $\tilde{X}/V(w_1)$ is isomorphic to the affine variety $\underset{\sim}{X} \subseteq \mathbb{C}^5$ cut out by the vanishing of the four entries of the product of braid matrices, minus the identity, given by

$$B_1(\zeta)B_1(z_3)B_1(z_4)B_1(z_5)B_1(z_6) - Id.$$

This precisely describes is the affine spectrum Spec $H^0(\mathcal{A}(\vartheta))$ *of the cohomology $H^*(\mathcal{A}(\vartheta)) = H^0(\mathcal{A}(\vartheta))$ of the DG-algebra $\mathcal{A}(\vartheta)$ associated to the positive braid word $\vartheta = \sigma_1^5$, which is readily equivalent to the braid word β. In this case, the degree -1 generator w_1 of β contributed to the vector field $V(w_1)$ tangent to $\mathcal{H}^0(\mathcal{A}(\beta))$. In general, there will be as many (commuting) vector fields freely acting on $\mathcal{H}^0(\mathcal{A}(\beta))$ as there are negative crossings in β.* □

Note that, in general, the introduction of Reidemeister II moves will require the use of *immersed*, and not just embedded, regions. For instance, we might have varied Example 2.12 by considering the braid word $\beta = \sigma_1^3\sigma_1^{-3}\sigma_1 \in \mathcal{B}_2$ or the braid word $\beta = \sigma_1\sigma_1^{-1}\sigma_1^2\sigma_1^{-2}\sigma_1 \in \mathcal{B}_2$. In both these cases, there are (orientable) immersed regions contributing to the differentials which are *not* embedded regions. Two such regions are depicted in Figure 12.

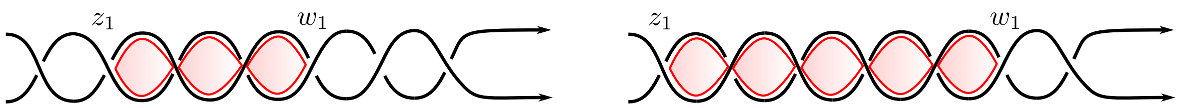


FIGURE 12. Immersed regions, both orientable, contributing to the differentials ∂z_1 for the braid word $\beta = \sigma_1^3\sigma_1^{-3}\sigma_1 \in \mathcal{B}_2$, on the left, and for the braid word $\beta = \sigma_1\sigma_1^{-1}\sigma_1^2\sigma_1^{-2}\sigma_1 \in \mathcal{B}_2$, on the right. In both cases, the red regions indicate immersed disks that yield the terms w_1 in ∂z_1.

2.4. **Proof of Theorem 1.5.** Let $\eta \in \mathcal{B}_n$ be such that $[\eta] \in \mathrm{Br}_n^+$ and let $\beta := \eta\Delta_n$. In this section we show that the braid pair $(X(\vartheta), V(\vartheta))$ is an invariant of the braid word η up to Δ-equivalence, as described in Theorem 1.5. In precise terms, we prove that the affine isomorphism type of the quotient $X(\eta)/V(\eta)$ remains invariant under Reidemeister II moves, Reidemeister III moves, and Δ-conjugations, and it is multiplied or quotiented by a trivial $\mathbb{C}^*$-factor under positive stabilizations and destabilizations.

Reidemeister III Moves. The Reidemeister move $\sigma_i\sigma_{i+1}\sigma_i \leftrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}$ between positive crossings preserves the affine variety $X(\eta)$ thanks to the identity

$$B_i(z_1)B_{i+1}(z_3)B_i(z_2) = B_{i+1}(z_2)B_i(z_3 - z_1z_2)B_{i+1}(z_1).$$

Namely, under a Reidemeister III move $\sigma_i\sigma_{i+1}\sigma_i \leftrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}$, the initial variables z_1, z_2, z_3 for the three crossings on the left, as Figure 13, relate to the resulting variables z'_1, z'_2, z'_3 on the right via

$$z_1 \longmapsto z'_2, \quad z_2 \longmapsto z'_1, \quad z_3 \longmapsto z'_3 + z'_1 z'_2.$$

This change of variables readily preserves the differentials of z_1, z_2: Figure 13 illustrates the case of z_2, where the regions contributing to ∂z_2 and $\partial z'_2$ are drawn, with colors indicating the bijection. Hence, the Reidemeister III does not affect the vector fields just involving z_1, z_2. For z_3, the change of variables $z'_3 + z'_1 z'_2$ is precisely such that the isomorphism is of DG-algebras. This follows by comparing regions corresponding to the differential ∂z_3 and its image

$$\partial z_3 \longmapsto \partial(z'_3 + z'_1 z'_2) = \partial z'_3 + \partial(z'_1 z'_2) = \partial z'_3 + \partial(z_2 z_1) = \partial z'_3 + z_1\partial(z_2) + z_2\partial(z_1).$$

For instance, the crossing z'_3 has a contribution in its differential of the form $z'_2 \cdot \mathfrak{c}'$, where $\mathfrak{c}'$ is a term that would be obtained by following the two bottom strands to the right. The region giving this contribution is depicted in blue in Figure 13 (bottom right), and it is precisely $\mathfrak{c}' = (\partial z'_1)_+$, the terms in $\partial(z'_1)$ counting regions to the right of z'_1. In contrast, the crossing z_3 before performing the Reidemeister III move has a contribution in its differential of the form $z_2 \cdot \mathfrak{c}$, where $\mathfrak{c}$ is a term that would be obtained by following the two top strands to the right, as depicted in Figure 13 (bottom left). The term $\mathfrak{c}$ coincides with $(\partial z_1)_+$, counting regions to the right of z_1. By gathering all these regions together, and taking into account pieces to the right and the left of ∂z_3 and $\partial z'_3$, we conclude that the isomorphism intertwines with the differentials as defined. In particular, the braid variety $X(\eta)/V(\eta)$ is invariant under these Reidemeister III moves.

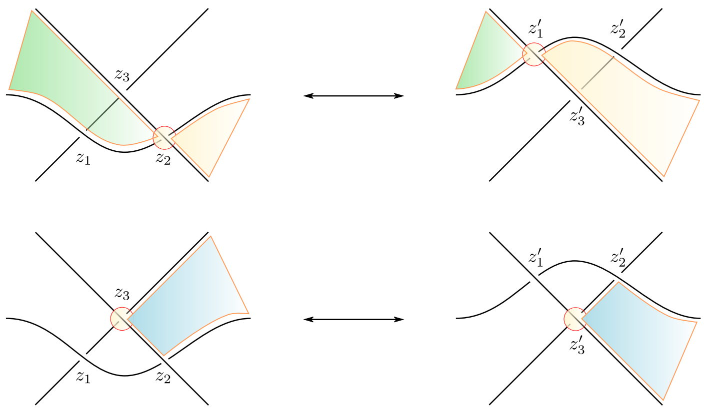


FIGURE 13. The Reidemeister III move and its effect on the differential of the degree 0 generators z_1, z_2, z_3. The regions are to be compared for the DG-algebra generated by z_1, z_2, z_3 and the DG-algebra generated by z'_1, z'_2, z'_3 under the isomorphism described in the text.

The argument for the Reidemeister move $\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} \leftrightarrow \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ between negative crossings also preserves the affine variety $X(\beta)$ thanks to the identity

$$B_i(0)B_{i+1}(0)B_i(0) = B_{i+1}(0)B_i(0)B_{i+1}(0).$$

An argument similar to the above, comparing the possible regions that contribute to the differential from degree 0 to degree 1, shows that these Reidemeister III moves yield DG-algebra isomorphisms.

Reidemeister II Moves. Let $\eta' \in \mathcal{B}_n$ be η with an additional canceling pair $\sigma_i\sigma_i^{-1}$ of crossings inserted, and let us write $\beta' = \eta'\Delta_n$. In comparison to $\mathcal{A}(\beta)$, the DG-algebra $\mathcal{A}(\beta')$ contains an additional pair of generators (z, w), in degree 0 and degree -1 respectively, such that the differential $\partial z = w + \ldots$ is linear on w, with coefficient one. This implies that z is a slice for the locally nilpotent derivation $V(w)$. In particular, the affine variety $X(\eta)$ is isomorphic to the intersection $X(\eta') \cap \{z = 0\}$, and the $\mathbb{C}$-action on $X(\eta')$ associated to $V(w)$ is free. (Note that $B_i(0)^2 = \mathrm{Id}$.) In addition, the affine variety $X(\eta')$ is isomorphic to $X(\eta') \cong X(\eta) \times \mathbb{C}$ and the required invariance follows. Note that (similarly to Example 2.12 and Figure 12) there is a bijection between the immersed disks connecting other pairs (z', w') before and after cancelling $\sigma_i\sigma_i^{-1}$, so all other vector fields transform correctly. The case where we insert $\sigma_i^{-1}\sigma_i$ is analogous.

In light of Proposition 2.3, we also need to check that the algebraic condition $H^{-1}(\mathcal{A}) = 0$ is preserved by the Reidemeister move. Indeed, assume that all generators $w_i \neq w$ of degree -1 are in the image of ∂. We get $\partial z = w + g(w_i)$ where g does not depend on w. By the assumption, $g(w_i)$ is a boundary, so that $g(w_i) = \partial x$ and $\partial(z - x) = w$. Therefore w is also in the image of ∂ and $H^{-1}(\mathcal{A}) = 0$.

Remark 2.13. In fact, the composition of moves $\sigma_i \to (\sigma_i\sigma_i^{-1})\sigma_i = \sigma_i(\sigma_i^{-1}\sigma_i) \to \sigma_i$, inserting a canceling pair $(\sigma_i\sigma_i^{-1})$ near a crossing σ_i and then canceling an adjacent pair, yields the identity isomorphism. □

Conjugation. Invariance of $(X(\sigma_i\eta), V(\sigma_i\eta))$ under the Δ-conjugation $\sigma_i\eta \to \eta\sigma_{n-i}$ is verified directly by studying the differentials in $\mathcal{A}(\beta)$. For the corresponding affine varieties $X(\sigma_i\eta)$ and $X(\eta\sigma_{n-i})$, we look at the differentials $\partial(y_{lm})$ on generators of degree 1, and we separately consider the comparison of the braid matrix terms

$$B_{\sigma_i\eta\Delta} \leftrightarrow B_{\eta\sigma_{n-i}\Delta},$$

and the corresponding Ш-terms. These latter terms involving $\text{Ш}(y_{lm})$ can be assumed to vanish up to a change of coordinates, as observed in Remark 2.8, and so we focus on the former braid matrix terms. Since $X(\sigma_i\eta)$ and $X(\eta\sigma_{n-i})$ are cut out by equations which equate $B_{\sigma_i\eta\Delta}$, resp. $B_{\eta\sigma_{n-i}\Delta}$ to a diagonal matrix, it is immediate that the affine varieties are invariant under such conjugation. Regarding the locally nilpotent derivations $V(\sigma_i\eta)$, $V(\eta\sigma_{n-i})$, we note that the differentials $\partial(z_j)$ are defined cyclically, up to a sign, due to the appearance of both terms $\mathcal{E}(B(\beta(z_j, w_k)); \iota(z_k), \iota(w_k))$ and $\mathcal{E}(B(\beta^c(z_j, w_k)); \iota(z_k), \iota(w_k))$ in $\partial(z_j)$. Thus, the regions contributing to $\partial(z_j)$ do not change if a crossing σ_i transfers from $\sigma_i\eta\Delta$ to $\eta\sigma_{n-i}\Delta$, and the associated vector fields yield isomorphic actions (and quotients). Note that σ_{n-i} appears for reasons strictly related to the braid group relation $\sigma_{n-i}\Delta_n = \Delta_n\sigma_i$, and not any further DG-algebra considerations.

Positive Stabilizations and Destabilizations. Let $\eta_n \in \mathcal{B}_n$, $\beta_n := \eta_n\Delta_n$, and denote by $\eta_{n+1} \in \mathcal{B}_{n+1}$ the *same* braid word as η_n, but considered as a braid word in $(n+1)$ strands. Let us consider $\eta_{n+1}\sigma_n \in \mathcal{B}_{n+1}$ and associate the degree 0 generator z in $\mathcal{A}(\eta\sigma_n\Delta_{n+1})$ to this additional σ_{n+1}-crossing. We must compare the braid pairs $(X(\eta_n\Delta_n), V(\eta_n\Delta_n))$ and $(X(\eta_{n+1}\sigma_n\Delta_{n+1}), V(\eta_{n+1}\sigma_{n+1}\Delta_{n+1}))$. Given that the additional crossing σ_n is positive *and* the differential $\partial z = 0$ vanishes, it suffices to reduce to the case where $\eta \in \mathcal{B}_n^+$ is a positive braid word.

Let z_j be the degree 0 generators in $\mathcal{A}(\eta_n\Delta_n^2)$ associated to the positive crossings, $j \in [1, \ell(\eta_n) + 2\binom{n}{2}]$, and $\zeta_1, \dots, \zeta_n$ be the n generators associated to n crossings of the rightmost interval $\upsilon := \sigma_n\sigma_{n-1}\dots\sigma_2\sigma_1$ of Δ_{n+1}, i.e. those crossings of Δ_{n+1} *not* in Δ_n. The degree 0 generator associated to the rightmost σ_1-crossing in $\eta_{n+1}\Delta_{n+1}\sigma_1 \sim \eta_{n+1}\sigma_n\Delta_{n+1}$ will still be denoted by z.

In this case, the variety $X(\eta_n\Delta_n)$ is cut out by the condition that the matrix

$$B_{\eta_n}\left(z_1, \dots, z_{\ell(\eta_n)}\right) B_{\Delta_n}\left(z_{\ell(\eta_n)+1}, \dots, z_{\ell(\eta_n)+\binom{n}{2}}\right) w_{0,n}$$

is upper triangular. Let us denote by $m := (m_{ij})$ the $(n \times n)$-matrix associated to the factor

$$B_{\eta_n}\left(z_1, \dots, z_{\ell(\eta_n)}\right) B_{\Delta_n}\left(z_{\ell(\eta_n)+1}, \dots, z_{\ell(\eta_n)+\binom{n}{2}}\right),$$

so that the $(n+1) \times (n+1)$-matrix associated to the product

$$B_{\eta_{n+1}}\left(z_1, \dots, z_{\ell(\eta_n)}\right) B_{\Delta_n}\left(z_{\ell(\eta_n)+1}, \dots, z_{\ell(\eta_n)+\binom{n}{2}}\right) B_\upsilon(\zeta_1, \dots, \zeta_n)$$

reads as the product

$$\mathfrak{m} := \begin{pmatrix} m_{11} & \cdots & m_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & \zeta_n & \cdots & \zeta_1 \end{pmatrix} = \begin{pmatrix} 0 & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m_{n1} & \cdots & m_{nn} \\ 1 & \zeta_n & \cdots & \zeta_1 \end{pmatrix},$$

as it is readily verified that

$$B_v(\zeta_1,\dots,\zeta_n)=\begin{pmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ 1 & \zeta_n & \cdots & \zeta_1\end{pmatrix}.$$

The variety $X(\eta_{n+1}\Delta_{n+1}\sigma_1)$ is cut out by the $(n+1)\times(n+1)$ matrix condition that $(\mathfrak{m}\cdot B_1(z))\cdot w_{0,n+1}$ is upper triangular. It thus suffices to compute

$$\mathfrak{m}\cdot B_1(z)=\begin{pmatrix}0 & m_{11} & \cdots & m_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & m_{n1} & \cdots & m_{nn}\\ 1 & \zeta_n & \cdots & \zeta_1\end{pmatrix}\begin{pmatrix}0 & 1 & \cdots & 0\\ 1 & z & \ddots & \vdots\\ \vdots & \vdots & \cdots & 0\\ 0 & 0 & \cdots & 1\end{pmatrix}=\begin{pmatrix}m_{11} & z\cdot m_{11} & \cdots & m_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n1} & z\cdot m_{n1} & \cdots & m_{nn}\\ \zeta_n & 1+z\zeta_n & \cdots & \zeta_1\end{pmatrix}$$

and note that, by direct comparison, $(\mathfrak{m}\cdot B_1(z))w_{0,n+1}$ is upper triangular if and only if $m\cdot w_{0,n}$ is upper triangular *and* the bottom row $(\zeta_n, 1+z\zeta_n,\cdots,\zeta_1)$ of $(\mathfrak{m}\cdot B_1(z))w_{0,n+1}$ is parallel to $(1,0,\cdots,0)$, i.e. $\zeta_i=0$, for $i\in[1,n-1]$, and $1+z\zeta_n=0$. The equation $1+z\zeta_n=0$ allows us to determine ζ_n in terms of z, and this latter z-variable is non-vanishing $z\in\mathbb{C}^*$. Hence we obtain the required isomorphism

$$X(\eta_{n+1}\sigma_n\Delta_{n+1})\cong X(\eta_n\Delta_n)\times\mathbb{C}^*_z.$$

Adding a disjoint strand. We keep the notation from the previous paragraph, so $\eta_n\in\mathcal{B}_n$ and $\eta_{n+1}\in\mathcal{B}_{n+1}$ is the same braid word as η_n, considered as a braid word in $(n+1)$ strands. We want to compare the braid pairs $(X(\eta_n\Delta_n),V(\eta_n\Delta_n))$ and $(X(\eta_{n+1}\Delta_{n+1}),V(\eta_{n+1}\Delta_{n+1}))$. Since we are not adding any extra crossing to η_n it suffices, just as in the previous check, to consider the case where $\eta_n\in\mathcal{B}_n^+$. Using the notation from above, we need the matrix $\mathfrak{m}w_{0,n+1}$ to be upper triangular. But it is immediate that this is the case if and only if the matrix $m=(m_{ij})$ is upper triangular and $\zeta_1=\cdots=\zeta_n=0$. Thus, we obtain

$$X(\eta_{n+1}\Delta_{n+1})\cong X(\eta_n\Delta_n).$$

This concludes the proof of Theorem 1.5. It should be noted that, conceptually, a neat proof of Theorem 1.5 is also brought forth by contact topology, as will be discussed in Subsection 2.5. In brief, it can be proven that the DG-algebra $\mathcal{A}(\beta)$ is the Legendrian contact DG-algebra of the Legendrian link associated to the Lagrangian (-1)-closure of the braid β, and thus its cohomology $H^0(\mathcal{A}(\beta))$ is a Legendrian invariant, from which Theorem 1.5 would follow by Floer geometric means. Instead, we have preferred to maintain our present arguments within the algebra realm and prioritized giving a direct proof of invariance. The reader interested in the connection with contact topology can proceed to the next subsection.

2.5. **Legendrian Links and Symplectic Topology.** Let us conclude this section by discussing the contact and symplectic topology that steers part of the results of the manuscript, including Theorem 1.5. The ambient geometry is given by the standard contact 3-space $(\mathbb{R}^3,\xi_{\rm st})$ and a piece of its symplectization $\mathbb{R}^3\times\mathbb{R}$, which is symplectomorphic to the standard Darboux 4-ball $(\mathbb{D}^4,d\lambda_{\rm st})$. The contact boundary $(\partial\mathbb{D}^4,\ker\lambda_{\rm st}|_{\partial\mathbb{D}^4})$ is contactomorphic to the standard contact 3-sphere $(\mathbb{S}^3,\xi_{\rm st})$, and the complement of a point in this latter contact 3-manifold is contactomorphic to $(\mathbb{R}^3,\xi_{\rm st})$. We implicitly interchange $(\mathbb{R}^3,\xi_{\rm st})$ and $(\mathbb{S}^3,\xi_{\rm st})$ as any objects we discuss can be assumed to lie on the complement of a fixed point. See [4, 36] for the basics of contact topology and [12, 21, 22] for details on Legendrian links.

Let us fix a Legendrian link $\Lambda\subseteq(\mathbb{R}^3,\xi_{\rm st})$, then the intuitive geometric question that gives rise to the algebraic objects being discussed, such as the braid variety, or the Richardson and positroid strata, is the following: what is the moduli space of embedded exact Lagrangian surfaces $L\subseteq(\mathbb{D}^4,d\lambda_{\rm st})$ in the standard symplectic 4-ball whose intersection $\partial L=L\cap\partial\mathbb{D}^4\subseteq(\mathbb{R}^3,\xi_{\rm st})$ is $\partial L=\Lambda$?

Such Lagrangian surfaces are said to be *Lagrangian fillings* of the Legendrian link Λ. This question needs to be phrased carefully if one is to be precise. First, we would like to understand the set of embedded exact Lagrangian surfaces $L\subseteq(\mathbb{D}^4,d\lambda_{\rm st})$ which have a fixed conical end of the form $\Lambda\times(-\varepsilon,0]$, for some small $\varepsilon\in\mathbb{R}^+$, i.e. in a neighborhood of ∂L, the surface is fixed to be the product $\Lambda\times(-\varepsilon,0]$. Second, a few degenerations of exact Lagrangian surfaces $L\subseteq(\mathbb{D}^4,d\lambda_{\rm st})$ are allowed, such as certain immersed Lagrangian filling. Third, each Lagrangian fillings will be endowed with a $\mathrm{GL}_1(\mathbb{C})$-local system when considering their moduli. And fourth, we are interested in studying Lagrangian fillings *up to* compactly supported Hamiltonian diffeomorphism of the standard 4-ball $(\mathbb{D}^4, d\lambda_{\mathrm{st}})$. In contact topology, a candidate for such a moduli space of Lagrangian fillings up to Hamiltonian isotopy is constructed as follows.

Given the fixed Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, we consider the Legendrian contact DG-algebra $\mathcal{C}(\Lambda)$. This DG-algebra was first introduced by Y. Chekanov [15], and see also [12, 21, 22] for a survey, and it has two salient properties:

Theorem 2.14 ([15, 19])**.** *Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian link and $\mathcal{C}(\Lambda)$ its Legendrian contact algebra. The following holds:*

(i) $\mathcal{C}(\Lambda)$ is a Legendrian isotopy invariant, i.e. if $\Lambda' \subseteq (\mathbb{R}^3, \xi_{st})$ is Legendrian isotopic to Λ, then the DG-algebra $\mathcal{C}(\bar{\Lambda})$ is stable tame isomorphic to $\mathcal{C}(\Lambda')$. In particular, the cohomology $H^*(\mathcal{C}(\Lambda)) \cong H^*(\mathcal{C}(\Lambda'))$ remains invariant.

(ii) Each embedded exact Lagrangian filling of Λ induces an augmentation

$$\varepsilon_L : \mathcal{C}(\Lambda) \longrightarrow \mathbb{Z}[H_1(L, \mathbb{Z})],$$

i.e. a DG-algebra morphism to the trivial DG-algebra $R = \mathbb{Z}[H_1(L, \mathbb{Z})]$, concentrated in degree 0 and with trivial differential.[5]

In contact topology, Theorem 2.14.(ii) indicates that a rigorous Floer theoretic avatar of the *moduli of Lagrangian fillings of* Λ is the set $\mathcal{M}(\Lambda)$ of all R-augmentations of $\mathcal{C}(\Lambda)$. It can be proven that $\mathcal{M}(\Lambda)$ is an affine R-scheme, and it has become one of the central moduli space of interest in contact topology. The quest for understanding geometric properties of the affine varieties $\mathcal{M}(\Lambda)$, such as partial compactifications, the existence of holomorphic symplectic structures, cluster structures and so forth, is an interesting one, e.g. see [11, 35]. There is no known algebraic geometric characterization for those affine varieties which arise as $\mathcal{M}(\Lambda)$. Nevertheless, several remarkable families of known moduli spaces and classical spaces, such as Richardson varieties and wild character varieties, appear as $\mathcal{M}(\Lambda)$ for certain classes of Legendrian links Λ. Now comes the simple, yet crucial, question: is there a way to associate a Legendrian link $\Lambda(\beta)$ to a braid word $\beta \in \mathcal{B}$ such that equivalent $\beta \sim \beta'$ yield Legendrian isotopic links $\Lambda(\beta) \simeq \Lambda(\beta')$?

If we were able to do so, we would obtain a DG-algebra associated to any braid word $\beta \in \mathcal{B}_n$ and the affine variety $\mathcal{M}(\Lambda(\beta))$ would be an invariant of the braid word β. The short answer to the question is that *it depends.* For a *positive* braid word $\beta \in \mathcal{B}^+$, we will momentarily describe a Legendrian link $\Lambda(\beta\Delta^2) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ with these properties. In this case, $\mathcal{M}(\Lambda(\beta\Delta^2))$ is precisely the braid variety $X(\beta\Delta; w_0)$, which we studied in [11], and its cohomology $H^*(\mathcal{M}(\Lambda(\beta\Delta^2)))$ is tightly related to the *link homology* of the smooth link underlying $\Lambda(\beta\Delta^2)$. In contrast, for a sufficiently negative braid word β, such as $\beta = \sigma_1^{-3} \in \mathcal{B}_2$ or a braid obtained by a *negative* Markov stabilization, there is no known way (and in a sense, there cannot be) to associate a Legendrian link $\Lambda(\beta)$ to β with the basic invariance properties. The threshold for which braid words are allowed is, nevertheless, a bit more interesting than just the positive–or–not dichotomy.

Let us now discuss how to construct Legendrian links in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ associated to braid words $\beta \in \mathcal{B}$. Let us choose Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$ and the contact form $\alpha_{\mathrm{st}} = dz - ydx$, such that $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$. These Legendrian links will be described by using either front projections, which describe their image in $\mathbb{R}^2_{x,z}$ under the Legendrian projection $\pi_{x,z} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2_{x,z}$ given by $\pi_{x,z}(x, y, z) = (x, z)$, or Lagrangian projections, which describe their image in $\mathbb{R}^2_{x,y}$ under the projection $\pi_{x,y} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2_{x,y}$ given by $\pi_{x,y}(x, y, z) = (x, y)$. We can proceed with either:

Legendrian Fronts. A crossing in the front projection always lifts to the *same* type of crossing, a positive one. Thus, positive braids $\delta, \vartheta \in \mathcal{B}^+$ can be drawn in $\mathbb{R}^2_{x,z}$, as a usual link tangle, and they define a Legendrian tangle. These tangles can be closed up in different ways: the two mains ones

[5] Base point t-variables and the choice of spin structures are required in order to have a $\mathbb{Z}$-algebra, and not just a $\mathbb{Z}_2$-algebra, but we will refer to [12] for these technical details.

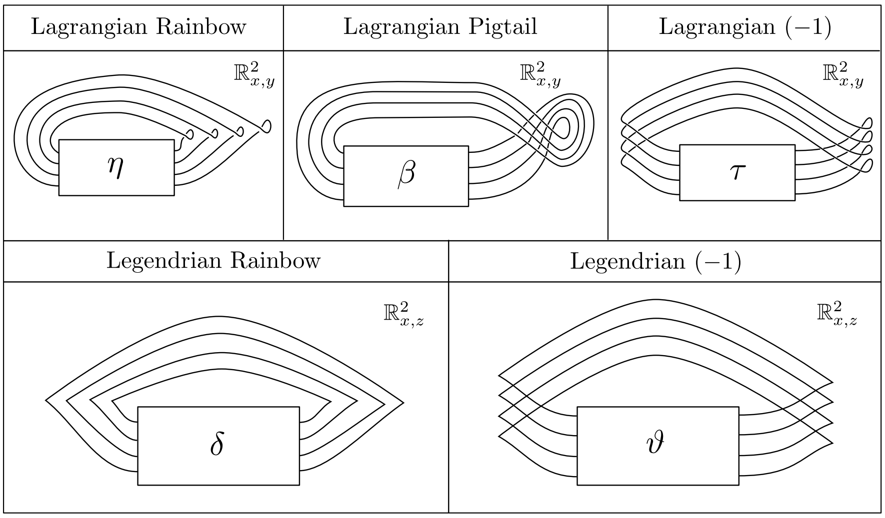


FIGURE 14. Three possible closures of braids in the Lagrangian projection, where $\eta, \beta, \tau \in \mathrm{Br}_n$ are not necessarily positive braids (First Row). Two possible closures of braids in the Legendrian projection, where $\delta, \vartheta \in \mathrm{Br}_n^+$ must be positive braids (Second Row).

are the Legendrian rainbow closure of a positive braid word δ, and the Legendrian (-1)-closure of a positive braid word ϑ. They are depicted in the bottom row of Figure 14. There is a relation between them: the Legendrian link given by the rainbow closure of δ is Legendrian isotopic to the Legendrian link given by the Legendrian (-1)-closure of $\delta\Delta^2$.

Lagrangian Projections. A crossing in the Lagrangian projection might be positive or negative, *but* it is *not true* that any knot diagram in $\mathbb{R}^2_{x,y}$ gives rise to a *Legendrian* knot in $(\mathbb{R}^3, \xi_{\mathrm{st}})$. Certain area considerations must be met for that to be true, and that makes the threshold interesting. First, for any *positive* braid words $\eta, \beta, \tau \in \mathcal{B}^+$, we can draw two closures in the Lagrangian projection: the Lagrangian rainbow closure of η and the Lagrangian (-1)-closure of τ, as depicted in the first row of Figure 14.

There is a relation between them: the Legendrian link given by the Lagrangian rainbow closure of η is Legendrian isotopic to the Legendrian link given by the Lagrangian (-1)-closure of $\eta\Delta^2$. It is also Legendrian isotopic to the Legendrian link given by the Legendrian rainbow closure of η, and thus to the Legendrian (-1)-closure of $\eta\Delta^2$.

There is a *third* Lagrangian projection that we can draw, associated to a braid word $\beta \in \mathcal{B}$: the Lagrangian Pigtail closure of β, as depicted in the middle of the top row of Figure 14. In general, it is *not* true that such a drawing defines a Legendrian link. For instance, if β is empty, this Lagrangian pigtail closure does *not* define a Legendrian link. A class of β for which the Lagrangian pigtail closure *does* yield a Legendrian link is given by the following:

Proposition 2.15 ([12]). *Let $\beta \in \mathcal{B}$ be a braid word of the form $\beta = \eta\Delta$, where $\eta \in \mathcal{B}$ is equivalent to a positive braid word $\eta_+ \in \mathcal{B}^+$ by a sequence of Reidemeister II moves, Reidemeister III moves and Δ-conjugations. Then the Lagrangian Pigtail closure of β represents a Legendrian link $\Lambda(\beta)$.*

In addition, the Legendrian link $\Lambda(\beta)$ is isotopic to the Lagrangian (-1)-closure of $\eta_+\Delta$. Also, if η_+ can be taken to be of the form $\eta_+ = \bar{\eta}_+\Delta$, $\bar{\eta}_+ \in \mathcal{B}^+$, $\Lambda(\beta)$ is Legendrian isotopic to the Lagrangian rainbow closure of $\bar{\eta}_+$. □

Geometrically, Proposition 2.15 is the contact topology result that allows us to go beyond the braid varieties associated to just positive braids, as we had considered in [11]. For completeness, let us state and succinctly prove the contact geometric theorem that implies Theorem 1.5, thus providing a more conceptual, independent second argument for Theorem 1.5. It reads as follows:

Theorem 2.16. *Let $\eta \in \mathcal{B}$ be a braid word which is equivalent to a positive braid word $\eta_+ \in \mathcal{B}^+$ by a sequence of Reidemeister II moves, Reidemeister III moves and Δ-conjugations. Consider the*

Legendrian link $\Lambda(\eta\Delta) \subseteq (\mathbb{R}^3, \xi_{st})$ associated to the Lagrangian Pigtail closure of $\eta\Delta$. Then the DG-algebra $\mathcal{A}(\eta\Delta)$ is isomorphic to the Legendrian contact DG-algebra $\mathcal{C}(\Lambda(\eta\Delta))$.

Proof. In the case that $\eta \in \mathcal{B}_+$ is a *positive* braid word, this was proven in [12], building on [49]. For the general case, where $\eta \in \mathcal{B}_n$ contains negative crossings, we need to justify the grading and, more importantly, the differentials. The generators of $\mathcal{C}(\Lambda(\eta\Delta))$ are given by crossings in the Lagrangian Pigtail of $\eta_n\Delta_n$, as they correspond to Reeb chords in the front projection. Hence, there are $\ell(\eta_n\Delta_n)+n^2$ generators.

The grading in $\mathcal{C}(\Lambda(\eta\Delta))$ is given by winding numbers[6], which are readily computed to be $1, 0$ and -1 for these Lagrangian projections. Indeed, the n^2 crossings in the pigtail yield generators y_{ml} of degree 1, because they are satellited from the 1-stranded case, where the unique generator bounds a closed loop of winding number 1. The positive crossings of $\eta\Delta$ give degree 0 generators z_j because the figure-eight Lagrangian projection of the standard Legendrian unknot – the Whitney immersion – has winding number 0. For the same reason, the negative crossings of $\eta\Delta$ give degree -1 generators w_k, since the winding number all around the braid is zero but the last $\pi/4$-rotation to match up the strands contributes to a resulting -1 winding number. See for example [21, 22] for more details on such grading computations.

The differential in $\mathcal{C}(\Lambda(\eta\Delta))$ counts rigid pseudo-holomorphic strips with asymptotic ends on Reeb chords of $\Lambda(\eta\Delta)$, which in this case translate into counting certain allowed regions in the Lagrangian Pigtail projection bounded by the strands and the crossings. We refer to [12, 22] for a combinatorial description of such allowed regions. The differentials ∂w_k vanish for degree reasons. The differentials ∂y_{lm}, ∂z_j respectively, count polygons with a positive puncture at y_{ml}, respectively z_j, and arbitrarily many negative punctures on other degree 0 elements, respectively degree 1 elements.

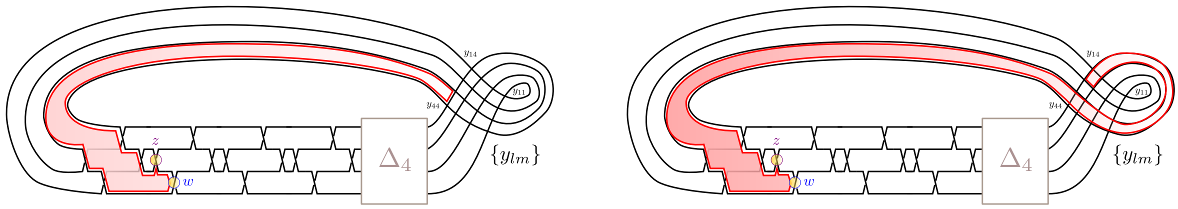


FIGURE 15. Two holomorphic strips contributing to the Ш-terms in the differential of the degree 1 generators for the DG-algebra $\mathcal{C}(\eta\Delta_4)$, for the braid word $\eta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_1^{-1}\sigma_3\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2 \in \mathcal{B}_4$. The red holomorphic strip on the left contributes to the coefficient $z \cdot w$ in the differential $\partial y_{34} = z(w \cdot y_{44})$, where z is of degree 0 and w of degree -1. The red holomorphic strip on the left contributes to the coefficient $z \cdot w$ in the differential $\partial y_{24} = z(w \cdot y_{23})$, where z is of degree 0 and w of degree -1.

For the differentials of the degree 1 generators, the regions contributing to the term $B_{\eta\Delta}$ in ∂y_{lm} are justified similarly to the case of a positive braid, the only new addition are possible punctures at negative crossings. The regions starting at the left positive quadrant of y_{lm}, and only using degree 0 and degree 1 negative punctures, cannot acquire a puncture at a negative crossing w_k. Indeed, their boundaries never come from the same side of w_k, and thus the only possibility at a negative σ_i^{-1}-crossing w_k is for the boundary of the region to switch strands; this is precisely accounted for in the matrix $B_i(0)$ and the totality of allowed regions at the left positive quadrant of y_{lm} is in bijection with the terms in the (l, m)-entry of $B_{\eta\Delta}$. In addition to the holomorphic strips contributing to $B_{\eta\Delta}$ in ∂y_{lm}, which consist solely of degree 0 and degree -1 generators, there may be holomorphic strips which start at either of the positive quadrants of y_{lm} – left or right – and use exactly one degree 1 generator and exactly one degree -1 generator. Two such strips are depicted in Figure 15. By dimension count, any rigid strip with a positive puncture at y_{lm} and exactly one negative puncture at w (with possibly more negative punctures at degree 0 generators) must be using exactly one degree 1

[6]Given an appropriate choice of capping paths in the case of a link.

generator y_{pq}. The grid combinatorics of the n^2 crossings in the pigtail imply that the generator y_{pq} must be in either the lth strand or the mth strand. Should y_{pq} be used, then the rigid strip is forced to go parallel between two strands until it reaches the region with the braid word $\eta\Delta$. If it reaches the braid word from the right, then the presence of Δ_4 forbids from any pair strands hitting any crossing – in particular a negative crossing – to the left of Δ_4. Thus those strips do not exist and only y_{pq} which lie *below* the generator y_{lm} may contribute to ∂y_{lm}. It is readily seen that the contributions are given by those regions described by $Ш(y_{lm})$, for the same combinatorial reason that the term $B_{\eta\Delta}$ accounted for the initial term of the ∂y_{lm}.

Finally, for the differentials ∂z_j of the degree 0 generators, note that the positive quadrants are to the right and left of the corresponding crossings. By degree reasons, any rigid strip that has a positive puncture at z_j cannot contain any degree 1 generators. Thus, strips may emerge from the right or the left of z_j, potentially continue through the pigtail without *any* punctures there, and proceed to the other side of the braid word $\eta\Delta$. These holomorphic strips are in bijection with the terms account for in the two coefficients $\mathcal{E}(B(\beta(z_j, w_k)); \iota(z_k), \iota(w_k))$, $\mathcal{E}(B(\beta^c(z_j, w_k)); \iota(z_k), \iota(w_k))$, one of which records the rigid strips with a positive puncture in the left quadrant of z_j and the other records the rigid strips with a positive puncture in the right quadrant of z_j. □

3. Positroid Braids and Proof of Theorem 1.3

In this section we introduce the Richardson braid $R_n(u, w)$, the juggling braid and the Le braid, and prove Theorem 1.3.(i). This occupies the majority of this section. Then, we discuss the matrix braid and conclude Theorem 1.3.(ii), which uses a much simpler argument. In particular, the results of this section show that the Legendrian links $\Lambda(R_n(u, w)\Delta_n^2)$, $\Lambda(J_k(f)\Delta_k)$ and $\Lambda(M_k(r))$ are Legendrian isotopic in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ if (u, w), f and r represent the same positroid content.

3.1. **Combinatorial Data.** Let us succinctly describe the combinatorial data that we use to describe the positroid braids. Fix two natural numbers $k, n \in \mathbb{N}$ such that $k \leq n$. There are four equivalent pieces of combinatorial objects, indexing open positroid strata, that we employ: certain pairs of permutations $u, w \in S_n$, certain bijections $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, Le diagrams and cyclic rank matrices. These are schematically depicted in Figure 16. The object of this subsection is to define part of these pieces of combinatorial data and review the bijections we will need.

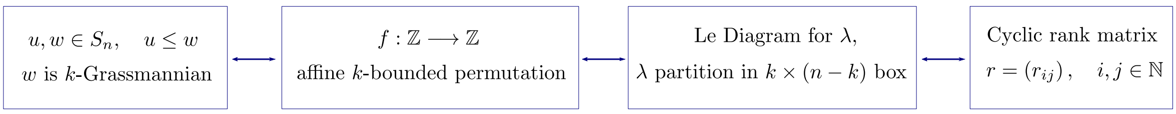


Figure 16. The four types of combinatorial data indexing positroid braids.

By definition, a permutation $w \in S_n$ is said to be k-Grassmannian if

$$w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k), \qquad \text{and } w^{-1}(k+1) < \cdots < w^{-1}(n).$$

A pair $u, w \in S_n$ such that $u \leq w$ in the Bruhat order and w is k-Grassmannian will be said to be a *positroid pair*. We will interchangeably discuss partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$, given by a weakly decreasing sequence of non-negative integers, and their associated Young diagrams, which we draw in French notation. We write $\lambda \subseteq (n-k)^k$ for partitions λ whose Young diagrams fit inside the $k \times (n-k)$ rectangle $(n-k)^k$. There exists a bijection between k-Grassmannian permutations $w \in S_n$ and Young diagrams $\lambda \subseteq (n-k)^k$, as recorded in [71, Section 19]. It is thus reasonable to denote w_λ for the permutations associated to the Young diagram λ. By using one-line notation, we can actually write

$$w_\lambda^{-1} = [1 + \lambda_k, 2 + \lambda_{k-1}, \ldots, k + \lambda_1, k + 1 - \lambda_1^t, k + 2 - \lambda_2^t, \ldots, n - \lambda_{n-k}^t],$$

and its length is $\ell(w_\lambda) = |\lambda| := \lambda_1 + \ldots + \lambda_{n-k}$. From a Young diagram λ, we read a reduced decomposition for the permutation w_λ as

$$\begin{aligned} w_\lambda = &\ (s_k s_{k+1} \cdots s_{k+\lambda_1-1})(s_{k-1} s_k \cdots s_{k+\lambda_2-2}) \cdots (s_1 \cdots s_{\lambda_k}) \\ = &\ (s_k s_{k-1} \cdots s_{k+1-\lambda_1^t})(s_{k+1} s_k \cdots s_{k+2-\lambda_2^t}) \cdots (s_{n-1} \cdots s_{n-\lambda_{n-k}^t}) \end{aligned},$$

where λ^t denotes the transposed Young diagram. This expression can be read pictorially: we draw the Young diagram λ and fill the box in row i and column j with the number $k+j-i$. The first reduced expression above is obtained by reading this diagram by rows, and the second reduced expression is obtained by reading it by columns.

Example 3.1. *Let us consider the values $k=3$ and $n=7$ and the Young diagram $\lambda=(4,3,1)$. By filling the (i,j)-box of λ with $3+(j-i)$, we obtain:*

1			
2	3	4	
3	4	5	6

Then, the associated 3-Grassmannian permutation is

$$w_\lambda = (s_3s_4s_5s_6)(s_2s_3s_4)(s_1) = (s_3s_2s_1)(s_4s_3)(s_5s_4)(s_6),$$

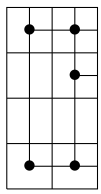
and note that the length $\ell(w_\lambda)$ is indeed $|\lambda| = 8$. □

Now, in order to additionally record the data of u in a positroid pair $u, w \in S_n$, one enhances the Young diagram λ for w into a *Le diagram.* By definition, a Le diagram is a collection of dots in λ such that the intersection of any pair of lines which belong to hooks for two different boxes *with a dot* must itself be in a box with a dot. It is proven in [71, Theorem 19.1] that considering the wiring diagram associated to a Le diagram for λ gives a bijection between Le diagrams and pairs $u, w_\lambda \in S_n$ such that $u \leq w_\lambda$ in Bruhat order. In particular, the boxes in the Le diagram with no dots correspond to the letters of u.

Example 3.2. *Let us consider the values $(k,n)=(4,6)$ and the Young diagram $\lambda=(2,2,2,2)$. The associated permutation is $w_\lambda = (s_4s_3s_2s_1)(s_5s_4s_3s_2)$. Choose the permutation $u=(s_3)(s_4s_2) \in S_6$, which readily satisfies $u \leq w$. The Le diagram associated to this pair (u,w) is drawn on Figure 17.(B).*

1	2
2	3
3	4
4	5

(A) The Young diagram $\lambda=(2,2,2,2)$ associated to $w_\lambda=(s_4s_3s_2s_1)(s_5s_4s_3s_2)$.



(B) The Le diagram associated to the pair (u, w_λ), for $u=(s_3)(s_4s_2)=(s_3s_2)(s_4)$.

FIGURE 17. Constructing a Le diagram from a pair (u,w).

Finally, let us discuss affine k-bounded permutations of size n, following [53]. By definition, an affine permutation $f:\mathbb{Z}\to\mathbb{Z}$ of size n is a bijection such that $f(i+n)=f(i)+n$ for all $i\in\mathbb{N}$; we often denote affine permutations in one-line window notation $f=[f(1)\dots f(n)]$. Also by definition, an affine permutation is said to be *k-bounded* if the following conditions are satisfied:

$$i \leq f(i) \leq i+n, \quad i\in\mathbb{N} \quad \text{and} \quad \sum_{i=1}^{n}(f(i)-i)=nk.$$

By [53], a k-bounded permutation f admits a unique decomposition of the form

$$f = u_f^{-1} t_k w_f, \qquad \text{where } t_k := [1+n, 2+n, \dots, k+n, k+1, k+2, \dots, n],$$

with $u_f, w_f \in S_n$ a positroid pair. Let us provide an explicit description of the permutations u_f, w_f appearing in this decomposition. For that, we note that there exists exactly k values $i_1 < i_2 < \cdots < i_k$ of $i\in[1,n]$ such that $n < f(i)$, and exactly $(n-k)$ values $j_1 < j_2 < \cdots < j_{n-k}$ of $i \in [1,n]$ such that $f(i) \leq n$. We describe u_f, w_f via their inverses u_f^{-1}, w_f^{-1}, which in one-line notation read

$$w_f^{-1} := [i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_{n-k}].$$

$$u_f^{-1} := [f(i_1) - n, \dots, f(i_k) - n, f(j_1), \dots f(j_{n-k})]$$

Note that w_f is a k-Grassmannian permutation and $u \in S_n$, since $n < f(i_r) \leq 2n$ for every $r \in [1, k]$. It is readily verified that u_f, w_f are the same permutations that are used in [53].

Example 3.3. *First, for the trivial n-translation $f = t_k = [1+n, \dots, k+n, k+1, \dots, n]$, we have $(i_1, \dots, i_k) = (1, \dots, k)$ and thus $w_f = [1, 2, \dots, n]$. Similarly, $u_f = [1, 2, \dots, n]$ is also the identity.*

Second, for the k-bounded permutation f defined by $f(i) = i + k$, we obtain that $(i_1, \dots, i_k) = (n-k+1, \dots, n)$ and hence $w = [k+1, \dots, n, 1, \dots, k]$ is the maximal k-Grassmannian permutation. In this second case, the permutation $u = [1, 2, \dots, n]$ is still the identity. □

Cyclic rank matrices, the fourth piece of combinatorial data, will be reviewed in Subsection 3.6. For now, we have enough ingredients to start addressing Theorem 1.3.(i), and we focus on that.

3.2. **Richardson Braid.** Given a permutation $v \in S_n$, we will denote by $\beta(v) \in \mathrm{Br}_n^+$ its positive braid lift to the n-stranded braid group. A positive braid word for $\beta(v) \in \mathrm{Br}_n^+$, which we also denote by $\beta(v) \in \mathcal{B}_n^+$, is obtained by considering a reduced expression $v = s_{i_1} \dots s_{i_{\ell(v)}}$ for $v \in S_n$ in terms of a product of the simple transpositions $s_1, \dots, s_{n-1} \in S_n$ generating the symmetric group S_n and substituting each s_i by the Artin braid generator σ_i, $i \in [1, n]$, i.e. $\beta(v) = \sigma_{i_1} \dots \sigma_{i_{\ell(v)}}$.

Definition 3.4 (Richardson Braid)**.** Let $u, w \in S_n$ be two permutations such that $u \leq w$ in the Bruhat order. The Richardson braid word $R_n(u, w) \in \mathcal{B}_n$ associated to the pair (u, w) is

$$R_n(u, w) := \beta(w)\beta(u)^{-1}.$$

By definition, the Richardson link is the Legendrian link $\Lambda(u, w) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, i.e. the Legendrian lift of the Lagrangian (-1)-closure of $R_n(u, w)\Delta_n^2$, cf. Definition 1.1. □

Note that the smooth type of the Richardson link $\Lambda(u, w)$ is that of the standard 0-framed (rainbow) closure of the braid word $R_n(u, w)$. Thus, the smooth link underlying $\Lambda(u, w)$ coincides with the smooth link studied in [33]. It is relevant to notice that the Richardson braid word $R_n(u, w) \in \mathcal{B}_n$ in Definition 3.4 is *not* necessarily a positive braid word, hence the need to consider the Lagrangian (-1)-closure in order to describe the Legendrian type of $\Lambda(u, w) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, if we are using the braid word $R_n(u, w)$ as an input. Note as well that $R_n(u, w)\Delta_n$ is equivalent to a positive braid word, and thus Proposition 2.15 applies.

In Theorem 4.3, we show that $X(R_n(u, w)\Delta_n)/V(R_n(u, w)\Delta_n)$ is isomorphic to the open Richardson variety $\mathcal{R}_u^w$ for any pair of permutations $u, w \in S_n$, and thus isomorphic to the open positroid stratum Π_u^w if w is k-Grassmannian. This is the reason for the terminology assigned to the braid word $R_n(u, w)$, [33]. A crucial feature of $R_n(u, w) \in \mathcal{B}_n$ is that it is a braid word in n-strands, whereas the juggling braid associated to (u, w) – defined momentarily – is a braid in k-strands. This dissonance in strands implies the necessity of using Markov moves when comparing the Richardson braid $R_n(u, w)$ to other positroid braids.

In order to address this, let us first show that the n-stranded braid word $R_n(u, w) \in \mathcal{B}_n$ is equivalent to a k-stranded braid word. Let us use the bijection between k-Grassmannian permutations w and Young diagrams λ, as explained in Subsection 3.1 above, and write the positive braid lift $\beta(w)$ of $w = w_\lambda$ as

$$\beta(w) = (\sigma_k \cdots \sigma_{k-\lambda_1^t+1})(\sigma_{k+1} \cdots \sigma_{k-\lambda_2^t+2}) \cdots (\sigma_{n-1} \cdots \sigma_{k-\lambda_d^t+d}) =$$
$$= \sigma_{[k-\lambda_1^t+1,k]}\sigma_{[k-\lambda_2^t+2,k+1]} \cdots \sigma_{[k-\lambda_d^t+d,n-1]},$$

where we fix the notation $d := n - k$ onwards. Recall that we are denoting interval braids by $\sigma_{[a,b]} := \sigma_b \cdots \sigma_a$, $a, b \in \mathbb{N}$, $a \leq b$. Similarly, write the positive braid lift $\beta(u)$ as

$$\beta(u) = u_1 \cdots u_d, \quad \text{where } u_i \subseteq \sigma_{[k+i-\lambda_i^t,k+i-1]}, \quad i \in [1, d].$$

We will momentarily use the following properties of braid groups:

(i) Let $a, c \in \mathbb{N}$, $a \leq c$, and consider a subword $u \subseteq \sigma_{[a,c]}$ of the interval braid word $\sigma_{[a,c]}$. Then the product $\sigma_{[a,c]}u^{-1}$ can be expressed as $\sigma_{[a,c]}u^{-1} = v^{-1}\sigma_{[b,c]}$, for some $b \in [a, c]$, where $v \subseteq \sigma_{[b,c-1]}$ is obtained by applying a sequence of Reidemeister III (and II) moves which, coarsely put, push u to the left. In precise terms, it suffices to do this for each crossing of u, and for a given crossing σ_k of u, we either use $\sigma_{[k,c]}\sigma_k^{-1} = \sigma_{[k+1,c]}$ or $\sigma_{[\ell,c]}\sigma_k^{-1} = \sigma_{k-1}^{-1}\sigma_{[\ell,c]}$,

$k \in [\ell+1, c]$. The braid word υ obtained in this manner, satisfying $\sigma_{[a,c]}u^{-1} = \upsilon^{-1}\sigma_{[b,c]}$, is said to be obtained by *sliding* the word u^{-1} through $\sigma_{[a,c]}$.

(ii) Let $a, b, c, d \in \mathbb{N}$ be such that $a \leq b \leq c \leq d$, then
$$\sigma_{[b,c]}\sigma_{[a,d]} = \sigma_{[a,d]}\sigma_{[b-1,c-1]} \quad \text{and} \quad \sigma^{-1}_{[b,c]}\sigma_{[a,d]} = \sigma_{[a,d]}\sigma^{-1}_{[b-1,c-1]}.$$
This is readily verified, and particularly immediate from a pictorial representation of the braids. We refer to this relation as a *nested interval exchange.*

We are ready to show that the n-stranded braid word $R_n(u, w) \in \mathcal{B}_n$ is equivalent to a k-stranded braid word. This is the content of the following

Proposition 3.5. *Let (u, w) be a positroid pair and $R_n(u, w) \in \mathcal{B}_n$ its associated Richardson word. Then there exist a d-tuple $(\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d$, a d-tuple of braid words $(v_1, \ldots, v_d) \in \mathcal{B}^d_k$, and a sequence of Reidemeister II and III moves and positive destabilizations which realize an equivalence between the n-stranded braid word $R_n(u, w)$ and the k-stranded braid*
$$v_d^{-1}\sigma_{[k-\gamma^t_d+1,k-1]}v^{-1}_{d-1}\sigma_{[k-\gamma_{d-1}+1,k-1]}\cdots v_2^{-1}\sigma_{[k-\gamma_2+1,k-1]}v_1^{-1}\sigma_{[k-\gamma_1+1,k-1]} \in \mathcal{B}_k,$$
where $\gamma_i \leq \lambda^t_i$ and $v_i \subseteq \sigma_{[k-\gamma_i+1,k-1]}$ for all $i \in [1, d]$.

Proof. Let us write $R_n(u, w) = \beta(w)\beta(u)^{-1}$ by using
$$\beta(w) = \beta(w_\lambda) = \sigma_{[k-\lambda^t_1+1,k]}\sigma_{[k-\lambda^t_2+2,k+1]}\cdots\sigma_{[k-\lambda^t_d+d,n-1]},$$
$$\beta(u) = u_1\cdots u_d, \quad \text{where } u_i \subseteq \sigma_{[k+i-\lambda^t_i,k+i-1]}, \quad i \in [1, d].$$
That is, the Richardson braid word reads
$$\beta(w)\beta(u)^{-1} = \sigma_{[k-\lambda^t_1+1,k]}\sigma_{[k-\lambda^t_2+2,k+1]}\cdots\sigma_{[k-\lambda^t_{d-1}+(d-1),n-2]}\sigma_{[k-\lambda^t_d+d],n-1}u_d^{-1}\cdots u_1^{-1}$$
where $u_i \subseteq \sigma_{[k+i-\lambda^t_i,k+i-1]}$, for all $i \in [1, d]$. The argument is now iterative, starting with u_d^{-1}. Since $u_d \subseteq \sigma_{[k+d-\lambda^t_d,k+d-1]} = \sigma_{[k+d-\lambda^t_d,n-1]}$, we can slide the word u_d^{-1} through $\sigma_{[k-\lambda^t_d+d,n-1]}$, producing the equivalent braid word
$$\beta(w)\beta(u)^{-1} = \sigma_{[k-\lambda^t_1+1,k]}\sigma_{[k-\lambda^t_2+2,k+1]}\cdots\sigma_{[k-\lambda^t_{d-1}+(d-1),n-2]}\overline{v_d}^{-1}\sigma_{[k-\gamma_d+d,n-1]}u^{-1}_{d-1}\cdots u_1^{-1},$$
where $\gamma_d \leq \lambda^t_d$ and $\overline{v_d} \subseteq \sigma_{[k-\gamma_d+d,n-2]}$ are defined according to the sliding process. In this expression, the crossing σ_{n-1} appears exactly once, in the term $\sigma_{[k-\gamma_d+d,n-1]}$. Thus, we can apply a positive Markov destabilization and write
$$\beta(w)\beta(u)^{-1} = \sigma_{[k-\lambda^t_1+1,k]}\sigma_{[k-\lambda^t_2+2,k+1]}\cdots\sigma_{[k-\lambda^t_{d-1}+(d-1),n-2]}\overline{v_d}^{-1}\sigma_{[k-\gamma_d+d,n-2]}u^{-1}_{d-1}\cdots u_1^{-1}.$$
First, we claim that $\overline{v_d}^{-1}$ can be pushed to the leftmost part of this braid word. Indeed, since $\overline{v_d} \subseteq \sigma_{[k-\gamma^t_d+d,n-2]}$ and $\gamma_d \leq \lambda^t_d \leq \lambda^t_{d-1}$, the word $\overline{v_d}$ can be first moved past $\sigma_{[n-2,k-\lambda^t_{d-1}+(d-1)]}$. The word $\sigma_{[k-\lambda^t_{d-1}+(d-1),n-2]}$ corresponding to the larger interval remains the same, and the former $\overline{v_d}$ yields a new word $\overline{v_d}'$, obtained by lowering the indices of $\overline{v_d}$ by one. Thanks to the inequalities $\gamma_d \leq \lambda^t_d \leq \lambda^t_i$ for all $i \in [1, d-1]$, and again using the hypothesis $\overline{v_d} \subseteq \sigma_{[k-\gamma_d+d,n-2]}$, the word $\overline{v_d}'$ can now be moved past the next interval braid. By iterating this process $(d-1)$ times, we obtain v_d which satisfies
$$\beta(w)\beta(u)^{-1} = v_d^{-1}\sigma_{[k,k-\lambda^t_1+1]}\sigma_{[k+1,k-\lambda^t_2+2]}\cdots\sigma_{[n-2,k-\lambda^t_{d-1}+(d-1)]}\sigma_{[n-2,k-\gamma_d+d]}u^{-1}_{d-1}\cdots u_1^{-1},$$
and $v_d^{-1} \subseteq \sigma_{[k-\gamma_d+1,k-1]}$, since it is obtained from $\overline{v_d}^{-1}$ by decreasing all the indices down by $(d-1)$. Second, we claim that the term $\sigma_{[k-\gamma_d+d,n-2]}$ in this latter expression for $\beta(w)\beta(u)^{-1}$ can also be pushed to the leftmost part of this braid word – leaving just v_d^{-1} to its left. This follows again from the inequalities $\gamma_d \leq \lambda^t_i$, which hold for all $i \in [1, d-1]$, and the interval in $\sigma_{[k-\gamma_d+d,n-2]}$ will always be the smaller one. Hence, all words to the left of $\sigma_{[k-\gamma_d+d,n-2]}$ will remain the same and the braid word $\sigma_{[k-\gamma^t_d+d,n-2]}$ will become $\sigma_{[k-\gamma_d+1,k-1]}$. This leads to the expression
$$\beta(w)\beta(u)^{-1} = \left(v_d^{-1}\sigma_{[k-\gamma_d+1,k-1]}\right)\left(\sigma_{[k-\lambda^t_1+1,k]}\sigma_{[k-\lambda^t_2+2,k+1]}\cdots\sigma_{[k-\lambda^t_{d-1}+(d-1),n-2]}u^{-1}_{d-1}\cdots u_1^{-1}\right),$$
where $\gamma_d \leq \lambda^t_d$ and $v_d \subseteq \sigma_{[k-\gamma_d+1,k-1]}$. The expression in red is precisely as required in the statement of Proposition 3.5, whereas the expression in blue is exactly of the same form than the expression at the start of this proof. Thus, we now iterate the same exact algorithm for the expression in blue,

which produces the d-tuples $(\gamma_1,\ldots,\gamma_d)\in\mathbb{Z}^d$ and $(v_1,\ldots,v_d)\in\mathcal{B}_k^d$ with the required properties. This concludes the argument. □

In the case of $u=1$ being the identity, the proof of Proposition 3.5 implies that the Richardson braid word $R_n(1,w)$ is equivalent to the k-stranded braid

$$\sigma_{[k-\lambda^t_{n-k}+1,k-1]}\cdots\sigma_{[k-\lambda^t_1+1,k-1]}$$

by a sequence of positive Markov stabilizations.

Example 3.6. *Let us choose the Young diagram $\lambda=(n-k)^k$ and $u=1$. Then the associated $w=w_\lambda$ is the maximal k-Grassmannian permutation in S_n, and $R_n(1,w)=\beta(w)$ is a shuffle braid. Smoothly, it yields the $(k,n-k)$-torus link. By Proposition 3.5, we may apply positive Markov destabilizations until we obtain the equivalent k-stranded braid word $(\sigma_{k-1}\cdots\sigma_1)^{n-k}$, which also yields to the $(k,n-k)$-torus link.*

Example 3.7. *Let us choose $k=4$ and $\lambda=(2,2,2,2)$; and thus $\lambda^t=(4,4)$. Then we have*

$$\beta(w)=(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2)\in\mathcal{B}_6$$

and $u=(\sigma_3)(\sigma_4\sigma_2)\le w$. The proof of Proposition 3.5 gives the following sequence of simplifications:

$$R_6(u,w)=\beta(w)\beta(u)^{-1}=(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2)(\sigma_2^{-1}\sigma_4^{-1})(\sigma_3^{-1})=$$

$$(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3)(\sigma_4^{-1})(\sigma_3^{-1})=(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_3^{-1})(\sigma_5\sigma_4\sigma_3)(\sigma_3^{-1})=$$

$$(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_3^{-1})(\sigma_4\sigma_3)(\sigma_3^{-1})=(\sigma_2^{-1})(\sigma_3\sigma_2)(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_3^{-1})=$$

$$(\sigma_2^{-1})(\sigma_3\sigma_2)(\sigma_2^{-1})(\sigma_4\sigma_3\sigma_2\sigma_1)=(\sigma_2^{-1})(\sigma_3\sigma_2)(\sigma_2^{-1})(\sigma_3\sigma_2\sigma_1)\in\mathcal{B}_4.$$

The two positive Markov destabilizations are marked in red in this sequence of equivalences. □

3.3. Juggling Braid.

In this subsection we construct the *juggling braid* $J_k(f)$ associated to an affine k-bounded permutation f and provide an algorithm to obtain a k-stranded positive braid word for $J_k(f)$.

Given a k-bounded affine permutation $f:\mathbb{Z}\longrightarrow\mathbb{Z}$, we consider the real plane $\mathbb{R}^2$ with Cartesian coordinates (x,y) and draw the integer values $1,2,\ldots,2n$ on the horizontal real x-axis $\{(x,y):y=0\}\subseteq\mathbb{R}^2$. For each of the values $i\in\mathbb{N}$, $1\le i\le n$, we draw the upper-circumference arc

$$A_i(f)=\{(x,y)\in\mathbb{R}^2:4(x-f(i)+1)^2+4y^2=(f(i)-i)^2\}\cap\{y\ge 0\}\subseteq\mathbb{R}^2$$

that starts at the point $(f(i),0)$ and ends at $(i,0)$. The union of these arcs is referred to as a *juggling diagram*, see [53] for the reason behind this terminology. By virtue of being a k-bounded affine permutation, there exist exactly k values $(i_1,\ldots,i_k)$ of i such that $n<f(i)\le 2n$. The juggling braid is defined via a tangle diagram obtained from the juggling diagram, as follows:

Definition 3.8 (Juggling Braid)**.** Let $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ be a k-bounded affine permutation of size n, and consider the k values $(i_1,\ldots,i_k)$ of i such that $n<f(i)\le 2n$. The juggling braid $J_k(f)$ associated to f is defined by the tangle diagram obtained by considering the union of the arcs $A_{i_1},\ldots,A_{i_k}\subseteq\mathbb{R}^2$ in the juggling diagram associated to f, declaring all crossings between these arcs to be *positive*, and smooth the intersections of the arcs with the x-axis, according to the local models in Figure 18. □

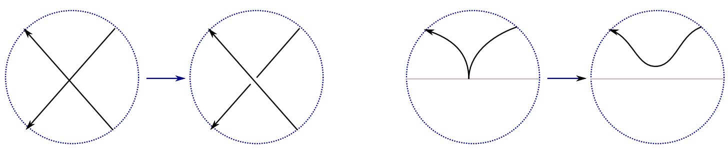
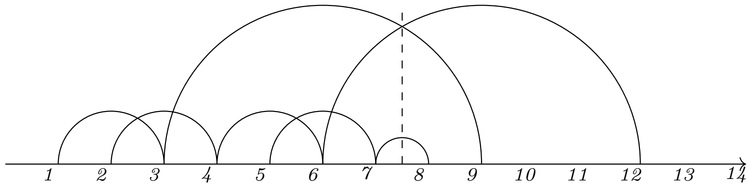


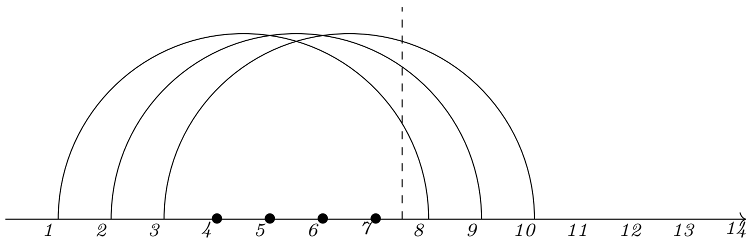
FIGURE 18. Local models constructing the juggling braid from the juggling diagram.

By construction, $J_k(f)$ is a positive k-stranded braid[7]. Note that there is a natural diagrammatic *closure* of this braid diagram, by connecting the points $f(i)$ and $f(i)-n$ by a lower-circumference arc below the axis if $n < f(i)$. The smooth link associated to this diagram is a closure for the braid $J_k(f)\Delta_k \in \mathcal{B}_k$, i.e. adding these lower-circumference arcs yields an additional Δ_k factor.

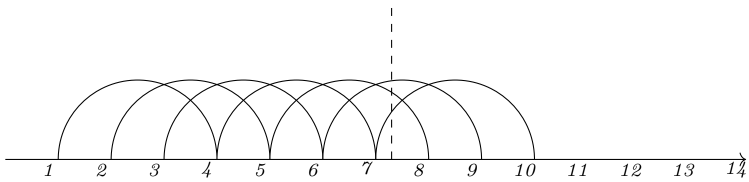
Example 3.9. *First, let us choose $k = 3, n = 7$ and $f = [3, 4, 9, 6, 7, 12, 8]$. Then the juggling diagram has the following form.*



Note that a braid word for it is $J_3(f) = \sigma_1\sigma_2\sigma_2\sigma_1\sigma_1$. Second, consider $k = 3, n = 7$ and $f = t_3 = [8, 9, 10, 4, 5, 6, 7]$; then the juggling diagram reads:



The braid $J_3(t_3)$ is the half twist in 3-strands. In general, the positive braid word $J_k(t_k)\Delta_k \in \mathcal{B}_k$ is the full twist $\Delta_k^2 \in \mathcal{B}_k$ on k-strands. Finally, as a third example, let us consider $k = 3, n = 7$ and $f = [4, 5, 6, 7, 8, 9, 10]$. The juggling diagram then reads:



More generally, if we consider the permutation $f(x) = x + k$, the corresponding links associated to the braids $J_k(f)\Delta_k$ will be (k, n)-torus links. □

Let $f = f(u, w) = u^{-1}t_k w$ be the k-bounded affine permutation associated to a positroid pair (u, w). The length of the braid word $J_k(f(u, w))$, which we also denote by $J_k(u, w)$, can be read from the pair (u, w) as follows.

Lemma 3.10. *Let $u, w \in S_n$ be a positroid pair. The length of $J_k(u, w)$ equals*

$$\ell(J_k(u, w)) = \ell(w) + \binom{k}{2} - \ell(u) - (n - k) + s$$

where s is the number of fixed points in the interval $[1, n]$ of the associated k-bounded affine permutation $f = u^{-1}t_k w$.

Proof. Let us denote $\beta_k := J_k(u, w)$ and suppose that $f = f(u, w)$ has no fixed points, which can be done without loss of generality. We use the notations as in Section 3.1. Two arcs starting at points x, y in the horizontal axis, $x < y$, will intersect if and only if $x < y < f(x) < f(y)$. Then, we have the following four cases depending of the values i_a, j_b that x, y acquire:

[7] In principle, one can use parabolas (which correspond to actual juggling trajectories) or other curves instead of circles in the definition of $J_k(f)$. As long as these curves are convex, the resulting braids are all related by Reidemeister III moves.

(i) $(x = i_a, y = j_b)$. Then $f(j_b) \leq n < f(i_a)$, so these arcs do not intersect.

(ii) $(y = i_b, x = j_a)$. Then the arcs intersect if $j_a < i_b < f(j_a)$, since $f(j_a) \leq n < f(i_b)$ automatically. The number of all pairs (j_a, i_b), $j_a < i_b$, equals $\ell(w)$, and we have to subtract $\sharp\{(a,b) : f(j_a) \leq i_b\}$.

(iii) $(x = i_a, y = i_b$ for $a < b)$. Then the arcs intersect if $f(i_a) < f(i_b)$, so (a,b) is not an inversion for u. The condition $i_b < f(i_a)$ is always satisfied.

(iv) $(x = j_a, y = j_b$ for $a < b)$. Then the arcs intersect if $j_b < f(j_a) < f(j_b)$. The number of such pairs equals

$$\binom{n-k}{2} - \sharp\{(a,b) : a < b, \quad f(j_b) < f(j_a)\} - \sharp\{(a,b) : a < b, \quad f(j_a) \leq j_b\}.$$

Observe that

$$\sharp\{(a,b) : f(j_a) \leq i_b\} + \sharp\{(a,b) : a < b, \quad j_b \leq f(j_a)\} = \sum_a (n - f(j_a) + 1),$$

and therefore the total number of intersection points equals

$$\ell(\beta_k) = \ell(w) - \sharp\{(a,b) : f(j_a) \leq i_b\} + \binom{k}{2} - \sharp\{(a,b) : a < b, \quad u(i_b) < u(i_a)\} +$$

$$\binom{n-k}{2} - \sharp\{(a,b) : a < b, \quad f(j_b) < f(j_a)\} - \sharp\{(a,b) : a < b, \quad f(j_a) \leq j_b\} =$$

$$= \ell(w) + \binom{k}{2} + \binom{n-k}{2} - \sharp\{(a,b) : a < b, \quad u(i_b) < u(i_a)\} -$$

$$-\sharp\{(a,b) : a < b, \quad u(j_b) < u(j_a)\} - \sum_a (n - u(j_a) + 1).$$

On the other hand, let us compute the number of inversions in u. The number of inversions involving i_b on the right equals $\sharp\{(a,b) : a < b, \quad u(i_a) > u(i_b)\}$. To compute the number of inversions involving j_b on the right, observe that there are $n - u(j_b)$ values greater than $u(j_b)$, and $\binom{n-k}{2} - \sharp\{(a,b) : a < b, \quad u(j_b) < u(j_a)\}$ of them are not inversions. Therefore

$$\ell(u) = \sharp\{(a,b) : a < b, \quad u(i_b) < u(i_a)\} + \sum_a (n - u(j_a)) +$$

$$-\binom{n-k}{2} + \sharp\{(a,b) : a < b, \quad u(j_b) < u(j_a)\}$$

and we conclude, for this case with $r = 0$, that

$$\ell(\beta_k) = \ell(w) + \binom{k}{2} - \ell(u) - (n-k).$$

It is immediate that adding fixed points increases the length by s and we have thus proven the required formula in the statement. □

Note that the difference $n - k$ which appears in the formula corresponds to the number of vertical cusps in the juggling diagram, i.e. to the right ends of short arcs connecting two points j_a and $f(j_a)$. In the comparison between the juggling braid $J_k(u,w)$ and the Richardson braid $R_n(u,w)$ we will use two additional results. First, an alternative algorithm that also computes the juggling braid $J_k(f)$, and a result that produces a positive k-stranded braid word for $J_k(f) = J_k(u,v)$ in terms of a Young diagram λ for $w = w_\lambda$ and the data of $u \leq w$. The next two subsections contribute with these two results.

3.3.1. *An algorithmic construction for $J_k(f)$.* Let us explicitly construct a braid word for the braid $J_k(f)$. First, note that the set $\{1,2,\dots,n\}\setminus\{f(1),f(2),\dots,f(n)\}$ has exactly k elements, which correspond precisely to the values $f(i_1)-n,\dots,f(i_k)-n$. Here is a recursive procedure to obtain a braid word for $J_k(f)$:

Algorithm 3.11. The input is a k-tuple of numbers $a=(a_1,\dots,a_k)$, the bounded affine permutation f, and a braid β.

Step 1: Choose i_0 such that $f(a_{i_0})=\min\{f(a_i)\mid i\in[1,k]\}$ and $a_{i_0}\in[1,n]$, and declare
$$j_0 := \max(\{j \mid a_j \le f(a_{i_0})\}\cap[1,n]).$$
Note that we have $i_0\le j_0$, since $i_0\le f(i_0)$.

Step 2: Declare
$$\beta' := \sigma_{[j_0-1,i_0]}\cdot\beta.$$
Note that $\sigma_{[j_0-1,i_0]}$ might be the trivial braid. Declare the new values of the k-tuple
$$a' := (a'_1,\dots,a'_k)$$
to be the elements in $(\{a_1,\dots,a_k\}\setminus\{a_{i_0}\})\cup\{f(a_{i_0})\}$ ordered such that $a'_1<a'_2<\ldots<a'_k$.

Step 3: If $\{a'_1<\dots<a'_k\}\cap[1,n]=\emptyset$, then print the braid β'.
Else, apply Algorithm 3.11 to the triple (a',f,β'). □

Lemma 3.12. *Let f be a k-bounded affine permutation, and consider the associated values $(i_1,\dots,i_k)$ in $[1,n]$ such that $n<f(i_j)$, $j\in[1,k]$. If Algorithm 3.11 is initialized with $\beta=1$ and $a_j:=f(i_j)-n$, $j\in[1,k]$, then it outputs a braid word for the juggling braid $J_k(f)$.*

Proof. The initialization numbers $a_1,\dots,a_k$ are precisely those points in the juggling diagram for f where the strands for $J_k(f)$, read from right to left, start. At each step of the algorithm, it produces precisely the crossings in the juggling diagram, as follows. Start with b_{i_0} to the left of the vertical dotted line
$$\{(x,y): 2x=2n+1\}\subseteq\mathbb{R}^2,$$
in the juggling diagram. Then, for each j with $b_j\le n$ and $b_{i_0}<b_j<f(b_{i_0})$ the strands containing b_{i_0} and b_j will cross. These are the crossings that the interval $\sigma_{j_0-1}\cdots\sigma_{i_0}$ produces. □

Example 3.13. *For instance, Algorithm 3.11 readily computes $J_k(1,w)$ where w is the k-Grassmannian permutation associated to a partition $\lambda\subseteq(n-k)^k$. Indeed, the affine permutation $f=t_kw$ has, in window notation, $(n+i)$ in position $(i+\lambda_{k-1+i})$, and $(k+j)$ in positions $(k+j-\lambda^t_j)$, $j\in[1,n-k]$, $i\in[1,k]$. Hence Algorithm 3.11 produces*
$$J_k(f)=(\sigma_1)(\sigma_2\sigma_1)\cdots(\sigma_{k-1}\cdots\sigma_1)(\sigma_{k-1}\cdots\sigma_{k+1-\lambda^t_{n-k}})\cdots(\sigma_{k-1}\cdots\sigma_{k+1-\lambda^t_2})(\sigma_{k-1}\cdots\sigma_{k+1-\lambda^t_1}).$$
This is to be compared with the Richardson braids at the beginning of Example 3.2 above. □

A simple modification of Algorithm 3.11 also produces an explicit braid word for the braid $J_k(f)\Delta_k$:

Algorithm 3.14. The input is a k-tuple of numbers $a=(a_1,\dots,a_k)$, the bounded affine permutation f, and a braid β.

Step 1: Choose i_0 such that $f(a_{i_0})=\min\{f(a_i)\mid i\in[1,k],\quad a_i\le n\}$ and $a_{i_0}\in[1,n]$, and declare
$$j_0 := \max(\{j \mid a_j\le f(a_{i_0})\}.$$

Step 2: Declare
$$\beta' := \sigma_{[j_0-1,i_0]}\cdot\beta.$$
Note that $\sigma_{[j_0-1,i_0]}$ might be the trivial braid. Declare the new values of the k-tuple
$$a' := (a'_1,\dots,a'_k)$$
to be the elements in $(\{a_1,\dots,a_k\}\setminus\{a_{i_0}\})\cup\{f(a_{i_0})\}$ ordered such that $a'_1<a'_2<\ldots<a'_k$.

Step 3: If $\{a'_1<\dots<a'_k\}\cap[1,n]=\emptyset$, then print the braid β'.
Else, apply Algorithm 3.14 to the triple (a',f,β'). □

In line with Lemma 3.12, Algorithm 3.14 initialized with $\beta = 1$ and $a_j := f(i_j) - n$, $j \in [1,k]$ prints a braid word for $J_k(f)\Delta_k$. Let us now discuss how to produce a positive k-stranded braid word for $J_k(f) = J_k(u,v)$ in terms of a Young diagram λ for $w = w_\lambda$ and the data of $u \le w$, where $u, w \in S_n$ are a positroid pair.

3.3.2. *Braid group action.* Let ${}^kS_n$ be the set of *left* k-Grasmannian permutations, i.e., $\mathtt{w} \in {}^kS_n$ if and only if $\mathtt{w}(1) < \cdots < \mathtt{w}(k)$, $\mathtt{w}(k+1) < \cdots < \mathtt{w}(n)$., and consider the set $(\mathbb{Z}^2)^n \times {}^kS_n$. Elements of $(\mathbb{Z}^2)^n \times {}^kS_n$ are to be thought of as triples $(a_i, b_i, \mathtt{w}(i))$, where $a_i, b_i \in \mathbb{Z}$, $i \in [1,n]$, and we will visually depict these elements as follows:

$$\underbrace{(a_1,b_1)}_{\mathtt{w}(1)}\underbrace{(a_2,b_2)}_{\mathtt{w}(2)}\cdots\underbrace{(a_n,b_n)}_{\mathtt{w}(n)}. \tag{3.1}$$

Associated to any such an element $(\vec{a}, \vec{b}, \mathtt{w}) = (a_i, b_i, \mathtt{w}(i))_i$, we introduce a braid

$$\sigma_{(\vec{a},\vec{b},\mathtt{w})} := \sigma_{[a_n,b_n]} \cdots \sigma_{[a_1,b_1]},$$

where $\sigma_{[a_i,b_i]}$ is an interval braid $\sigma_{b_i}\sigma_{b_i-1}\cdots\sigma_{a_i}$.[8] By definition, we say that a left k-Grassmannian permutation $\mathtt{w}$ *breaks* i if

$$\{\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1)\} \cap \{1,\dots,k\} \quad \text{and} \quad \{\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1)\} \cap \{k+1,\dots,n\}$$

are both nonempty, i.e. $\mathtt{w}$ breaks i if and only if $s_i\mathtt{w}$ is a left k-Grassmannian permutation. If $\mathtt{w}$ does not break i, we say that the pair $(i, i+1)$ is *on the left* of $\mathtt{w}$ if $\{\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1)\} \subseteq \{1,\dots,k\}$, and *on the right* of $\mathtt{w}$ if $\{\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1)\} \subseteq \{k+1,\dots,n\}$.

Let us now construct an action of the n-stranded braid group Br_n on the set $(\mathbb{Z}^2)^n \times {}^kS_n$. It suffices to define it on the Artin generators $\sigma_1, \dots, \sigma_{n-1}$, and we define $\sigma_i(\vec{a}, \vec{b}, \mathtt{w}) = (\vec{a}', \vec{b}', \mathtt{w}')$ according to the two following cases:

(i) ($\mathtt{w}$ does not break i). Then $(i, i+1)$ is either on the left or on the right of $\mathtt{w}$. In both cases, we define $\mathtt{w}' := \mathtt{w}$, and define σ_i such that it only changes the $\mathtt{w}^{-1}(i)$ and $\mathtt{w}^{-1}(i+1) = \mathtt{w}^{-1}(i)+1$ components of $(\vec{a}, \vec{b})$, as follows:

$$\underbrace{(a_j,b_j)}_{i}\underbrace{(a_{j+1},b_{j+1})}_{i+1} \overset{\sigma_i}{\mapsto} \begin{cases} \underbrace{(a_{j+1}+1,b_{j+1})}_{i}\underbrace{(a_j,b_j)}_{i+1} & \text{if } (i,i+1) \text{ is on the left of } w \\ \underbrace{(a_{j+1}+1,b_j)}_{i}\underbrace{(a_j,b_{j+1})}_{i+1} & \text{if } (i,i+1) \text{ is on the right of } w, \end{cases} \tag{3.2}$$

where we set $j := \mathtt{w}^{-1}(i)$.

(ii) ($\mathtt{w}$ breaks i). Then we define $\mathtt{w}' := s_i\mathtt{w}$, which is left k-Grassmannian. We also declare $a'_j := a_j$ and $b'_j := b_j$ except for the values of $j = \mathtt{w}^{-1}(i)$ and $\mathtt{w}^{-1}(i+1)$. In these two cases, we define a'_j, b'_j via:

$$\underbrace{(a_1,b_1)}_{\mathtt{w}(1)}\cdots\underbrace{(a_j,b_j)}_{i}\cdots\underbrace{(a_k,b_k)}_{i+1}\cdots\underbrace{(a_n,b_n)}_{\mathtt{w}(n)} \overset{\sigma_i}{\mapsto} \underbrace{(a_1,b_1)}_{\mathtt{w}(1)}\cdots\underbrace{(a_j,b_j)}_{i+1}\cdots\underbrace{(a_k,b_k)}_{i}\cdots\underbrace{(a_n,b_n)}_{\mathtt{w}(n)} \tag{3.3}$$

$$\underbrace{(a_1,b_1)}_{\mathtt{w}(1)}\cdots\underbrace{(a_j,b_j)}_{i+1}\cdots\underbrace{(a_k,b_k)}_{i}\cdots\underbrace{(a_n,b_n)}_{\mathtt{w}(n)} \overset{\sigma_i}{\mapsto} \underbrace{(a_1,b_1)}_{\mathtt{w}(1)}\cdots\underbrace{(a_j,b_j-1)}_{i}\cdots\underbrace{(a_k,b_k)}_{i+1}\cdots\underbrace{(a_n,b_n)}_{\mathtt{w}(n)} \tag{3.4}$$

Note that (3.3) changes only the permutation $\mathtt{w}$ but not the entries $\vec{a}, \vec{b}$.

Let us show that these formulas indeed define an action of the braid group Br_n:

Lemma 3.15. *Formulas* (3.2)*,* (3.3) *and* (3.4) *define an action of* Br_n *on* $(\mathbb{Z}^2)^n \times {}^kS_n$.

[8] Recall that we declare $\sigma_{[a_i,b_i]} = 1$ if it is not the case that $0 < a_i \le b_i$.

Proof. We only need to show the relation $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$. This is a tedious yet straightforward computation. First, if $\mathtt{w}$ does not break i nor $i+1$, then $(i, i+1)$ and $(i+1, i+2)$ are on the same side of $\mathtt{w}$. Then it is easy to check that part of interest transforms as follows under both $\sigma_i\sigma_{i+1}\sigma_i$ and $\sigma_{i+1}\sigma_i\sigma_{i+1}$.

$$\underbrace{(a_j, b_j)}_{i}\underbrace{(a_{j+1}, b_{j+1})}_{i+1}\underbrace{(a_{j+2}, b_{j+2})}_{i+2} \mapsto \underbrace{(a_{j+2}+2, b_{j+2})}_{i}\underbrace{(a_{j+1}+1, b_{j+1})}_{i+1}\underbrace{(a_j, b_j)}_{i+2}$$
$$\underbrace{(a_j, b_j)}_{i}\underbrace{(a_{j+1}, b_{j+1})}_{i+1}\underbrace{(a_{j+2}, b_{j+2})}_{i+2} \mapsto \underbrace{(a_{j+2}+2, b_j)}_{i}\underbrace{(a_{j+1}+1, b_{j+1})}_{i+1}\underbrace{(a_j, b_{j+2})}_{i+2}$$

where in the top line we assume that $(i, i+1)$ and $(i+1, i+2)$ are on the left of $\mathtt{w}$, and in the bottom line we assume that they are on the right. The remaining part can be verified by cases. In the first case, $\mathtt{w}$ does not break i but breaks $i+1$. In the second case, $\mathtt{w}$ does not break $i+1$ but breaks i, and in the third case $\mathtt{w}$ breaks both i and $i+1$. The proofs in these three cases are similar, and thus we provide the details for the first case, leaving the second and third cases as analogous exercises for the reader. In the case that $\mathtt{w}$ does not break i but breaks $i+1$, we proceed as follows.

If $\mathtt{w}$ does not break i but breaks $i+1$, the pair $(i, i+1)$ could be on the left, or on the right, of the permutation $\mathtt{w}$. In the former case, we directly compute:

$$\underbrace{(a_j, b_j)}_{i}\underbrace{(a_{j+1}, b_{j+1})}_{i+1}\cdots\underbrace{(a_k, b_k)}_{i+2} \overset{\sigma_i}{\mapsto} \underbrace{(a_{j+1}+1, b_{j+1})}_{i}\underbrace{(a_j, b_j)}_{i+1}\cdots\underbrace{(a_k, b_k)}_{i+2} \overset{\sigma_{i+1}}{\mapsto}$$
$$\underbrace{(a_{j+1}+1, b_{j+1})}_{i}\underbrace{(a_j, b_j)}_{i+2}\cdots\underbrace{(a_k, b_k)}_{i+1} \overset{\sigma_i}{\mapsto} \underbrace{(a_{j+1}+1, b_{j+1})}_{i+1}\underbrace{(a_j, b_j)}_{i+2}\cdots\underbrace{(a_k, b_k)}_{i}$$

and

$$\underbrace{(a_j, b_j)}_{i}\underbrace{(a_{j+1}, b_{j+1})}_{i+1}\cdots\underbrace{(a_k, b_k)}_{i+2} \overset{\sigma_{i+1}}{\mapsto} \underbrace{(a_j, b_j)}_{i}\underbrace{(a_{j+1}, b_{j+1})}_{i+2}\cdots\underbrace{(a_k, b_k)}_{i+1} \overset{\sigma_i}{\mapsto}$$
$$\underbrace{(a_j, b_j)}_{i+1}\underbrace{(a_{j+1}, b_{j+1})}_{i+2}\cdots\underbrace{(a_k, b_k)}_{i} \overset{\sigma_{i+1}}{\mapsto} \underbrace{(a_{j+1}+1, b_{j+1})}_{i+1}\underbrace{(a_j, b_j)}_{i+2}\cdots\underbrace{(a_k, b_k)}_{i}.$$

This indeed shows that the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is satisfied if $(i, i+1)$ is on the left of $\mathtt{w}$. If $(i, i+1)$ were on the right of $\mathtt{w}$, then it is equally straightforward to verify that the end result of applying both $\sigma_i\sigma_{i+1}\sigma_i$ and $\sigma_{i+1}\sigma_i\sigma_{i+1}$ is

$$\underbrace{(a_k, b_k-2)}_{i}\cdots\underbrace{(a_{j+1}+1, b_j)}_{i+1}\underbrace{(a_j, b_{j+1})}_{i+2}.$$

This concludes the argument. □

Let us now use this braid group action to produce a braid word for $J_k(u, w)$. The intuitive idea is that we start with the k-Grassmannian permutation w and consider a *positive* braid lift $\beta(u^{-1}) \in \mathrm{Br}_n$ of the word u^{-1}. We will momentarily define an element $(\vec{a}, \vec{b}, \mathtt{w}) \in (\mathbb{Z}^2)^n \times {}^{n-k}S_n$ so that the braid $\sigma_{\beta(u^{-1})(a,b,\mathtt{w})}$ associated to the resulting action of $\beta(u^{-1})$ applied to $(a, b, \mathtt{w})$ will coincide with the juggling braid $J_k(u, w)$. To that end, let us consider the Young diagram $\lambda \subseteq (n-k)^k$ associated to the k-Grassmannian permutation w and define the following element of $(\mathbb{Z}^2)^n \times {}^{n-k}S_n$:

$$\mathbf{x}_\lambda := \underbrace{(k-\lambda_1^t+1, k-1)}_{k+1}\underbrace{(k-\lambda_2^t+1, k-1)}_{k+2}\cdots\underbrace{(k-\lambda_{n-k}^t+1, k-1)}_{n}\underbrace{(1, k-1)}_{1}\underbrace{(1, k-2)}_{2}\cdots\underbrace{(1, 0)}_{k}.$$

Note that the permutation associated to $\mathbf{x}_\lambda$ is always the maximal left $(n-k)$-Grassmannian permutation. The vector $\mathbf{x}_\lambda$ gives us a bridge between the action of B_n on $(\mathbb{Z}^2)^n \times {}^{n-k}S_n$ and the braid word $J_k(u, w)$. Indeed, it follows from Example 3.13 that $J_k(1, w)$ is precisely the braid word associated to the vector $\mathbf{x}_\lambda$. Also, if we write $\mathbf{x}_\lambda = (\vec{a}, \vec{b}, \mathtt{w})$, note that $\mathtt{w}(1), \mathtt{w}(2), \ldots, \mathtt{w}(n-k), \mathtt{w}(n-k+1)+n, \ldots, \mathtt{w}(n)+n$ give us the new labels of the strands appearing in each step of Algorithm 3.14. The precise result reads:

Theorem 3.16. *Let $\lambda \subseteq (n-k)^k$ be a Young diagram, and $u \in S_n$ be such that $u \leq w_\lambda$. Then, the k-stranded braid $J_k(u, w_\lambda)$ is the braid associated to*

$$\beta(u^{-1}) \cdot (\mathbf{x}_\lambda),$$

where the dot indicates the image of $\mathbf{x}_\lambda$ acted upon by $\beta(u^{-1}) \in \mathrm{Br}_n$, a positive braid lift of the permutation u^{-1}.

Before giving the proof of Theorem 3.16, we need the following preparatory lemma.

Lemma 3.17. *In the same notation of Theorem 3.16, define $(\vec{a}, \vec{b}, \mathtt{w}) = \beta(u^{-1}) \cdot (\mathbf{x}_\lambda)$. Then*

$$\mathtt{w}(i+1) - \mathtt{w}(i) = b_{i+1} - b_i + 1, \quad i \in [1, n-k-1].$$

Proof. Let us proceed by induction on the length $\ell(u)$. In the base case $\ell(u) = 0$, the definition of $\mathbf{x}_\lambda$ implies $\mathtt{w}(i+1) - \mathtt{w}(i) = 1$ for $i \in [1, n-k-1]$, while $b_1, \ldots, b_{n-k} = k-1$. Thus we have $\mathtt{w}(i+1) - \mathtt{w}(i) = 1 = b_{i+1} - b_i + 1$ as needed.

To prove the induction step, we make the following observation: if we ignored the $\vec{a}, \vec{b}$ components in the action of Br_n on the set $(\mathbb{Z}^2)^n \times {}^{n-k}S_n$, we obtain an action of Br_n on ${}^{n-k}S_n$ which is transitive and factors through the symmetric group S_n. In fact, this action coincides with the one given by identifying the set of left $(n-k)$-Grassmannian permutations with the set of minimal-length left coset representatives of $S_{n-k} \times S_k$ in S_n. Let us be a bit more explicit about it. The stabilizer of the maximal left $(n-k)$-Grassmannian permutation $\mathtt{w}^{\max} \in {}^{n-k}S_n$ is $S_k \times S_{n-k}$. Moreover, if $\mathtt{w} \in {}^{n-k}S_n$ is any other $(n-k)$-Grassmannian permutation, a distinguished element $s_{\mathtt{w}} \in S_n$ satisfying $s_{\mathtt{w}}(\mathtt{w}^{\max}) = \mathtt{w}$ is

$$s_{\mathtt{w}} = (s_{\mathtt{w}(n-k)} \cdots s_{n-1}) \cdots (s_{\mathtt{w}(2)} \cdots s_k s_{k+1})(s_{\mathtt{w}(1)} \cdots s_{k-1} s_k).$$

It follows that if $v \in S_n$ is any permutation satisfying $v(\mathtt{w}^{\max}) = \mathtt{w}$, then $v^{-1} s_{\mathtt{w}} \in S_k \times S_{n-k}$.[9] In particular, if $v < s_i v$ – or equivalently, if $v^{-1}(i) < v^{-1}(i+1)$ – then we cannot have $s_{\mathtt{w}}^{-1}(i) \in \{k+1, \ldots, n\}$ and $s_{\mathtt{w}}^{-1}(i+1) \in \{1, \ldots, k\}$. By construction of $s_{\mathtt{w}}$, this is equivalent to saying that we cannot have $\mathtt{w}^{-1}(i) \in \{1, \ldots, n-k\}$ and $\mathtt{w}^{-1}(i+1) \in \{n-k+1, \ldots, n\}$. With this in mind, let us address the induction step for the proof of the lemma.

By induction, assume that the result is valid for a permutation u, and consider i such that $u^{-1} < s_i u^{-1}$. If both $\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1) \in \{1, \ldots, n-k\}$ then the induction assumption implies that the vector $\vec{b}$ remains unchanged after applying σ_i. Since the same is true for $\mathtt{w}$, the result follows in this case. If both $\mathtt{w}^{-1}(i)$ and $\mathtt{w}^{-1}(i+1)$ belong to $\{n-k+1, \ldots, n\}$, then again neither $\vec{b}$ nor $\mathtt{w}$ change. If $\mathtt{w}^{-1}(i+1) \in \{1, \ldots, n-k\}$ and $\mathtt{w}^{-1}(i) \in \{n-k+1, \ldots, n\}$ then, after applying σ_i, one of the components of $\vec{b}$ will decrease by 1; that said, the $\mathtt{w}$-component of the same index will also decrease by 1, as it changes from $i+1$ to i. From our discussion in the previous paragraph, applied with $v = u^{-1}$, we conclude that the last case, $\mathtt{w}^{-1}(i) \in \{1, \ldots, n-k\}$ and $\mathtt{w}^{-1}(i+1) \in \{n-k+1, \ldots, n\}$, cannot occur and the argument is concluded. $\square$

Finally, let us proceed with the proof of Theorem 3.16.

Proof of Theorem 3.16. Let us (also) proceed by induction on the length $\ell(u)$. In the base case $\ell(u) = 0$, the result follows from the definition of $\mathbf{x}_\lambda$ and Example 3.13. For the induction step, we assume the statement to hold for u, and we show that the result is also true for us_i, where $u < us_i \leq w$. In fact, we will prove a stronger statement: the new label appearing after applying the m-th interval braid $\sigma_{[a_m, b_m]}$ is precisely $\mathtt{w}(m)$. Denote the associated k-bounded affine permutation by $f := u^{-1} t_k w$, and note that the requirement that $w \geq us_i$ is equivalent to the fact that $s_i f$ is a k-bounded affine permutation. Let us prove this by considering cases.

First, let us assume that $i, i+1 \in f \cdot [1, n]$ and that none of them are fixed points of f. Then, if we write

$$\beta(u^{-1}) \mathbf{x}_\lambda = (\vec{a}, \vec{b}, \mathtt{w}),$$

we get that $\mathtt{w}^{-1}(i), \mathtt{w}^{-1}(i+1) \in \{1, \ldots, n-k\}$ and we have to apply the top row of (3.2) to obtain $(\sigma_i \beta(u)) \cdot (\mathbf{x}_\lambda)$. On the other hand, the part of the strand labeled by i in Algorithm 3.11 comes after

[9] In fact, note that $s_{\mathtt{w}}$ is left k-Grassmannian and the map $\mathtt{w} \mapsto s_{\mathtt{w}}$ gives an explicit bijection between the sets ${}^{n-k}S_n$ and ${}^{k}S_n$; we will nevertheless not need this.

applying the interval braid $\sigma_{[a_j,b_j]}$, and the part of the strand labeled by $i+1$ comes after applying the next interval braid $\sigma_{[a_{j+1},b_{j+1}]}$, where we set $j = \mathtt{w}^{-1}(i)$. Now, $(s_i f)^{-1}(i) = f^{-1}(i+1)$ implies that the new initial point of the interval finishing with label i will start at the position $a_{j+1}+1$. Similarly, the initial point of the interval finishing with label $i+1$ will start at a_j. Hence, we need to apply the transformation $\sigma_{[a_{j+1},b_{j+1}]}\sigma_{[a_j,b_j]} \mapsto \sigma_{[a_j,b_{j+1}]}\sigma_{[a_{j+1}+1,b_j]}$ to obtain the correct braid word. By applying Lemma 3.17, we obtain that $b_{j+1} = b_j$, and we see that the formula coincides with the top component of (3.2). If $i+1$ is a fixed point of f then $b_{j+1} < a_{j+1}$, and so $b_j < a_{j+1}+1$; thus we see that the braid action $(\sigma_i\beta)(\mathbf{x}_\lambda)$ indeed computes $s_i f$, which has i as a fixed point. Finally, if i is a fixed point of f then the condition $u < us_i$ forces $i+1$ to be a fixed point of f too. Otherwise, we would have $f^{-1}(i+1) < i < n$, and the formula for u in terms of f implies that $u(i+1) < u(i)$, which is a contradiction with the assumption that $u < us_i$.

The second case, when $n+i, n+i+1 \in f \cdot [1,n]$ is proven analogously. Finally, we arrive at the case where $i+1 \in f\cdot[1,n]$, and $n+i \in f\cdot[1,n]$. Note that the case $i \in f\cdot[1,n]$ and $n+i+1 \in f\cdot[1,n]$ is excluded by the condition $u < us_i$. In this case, $\mathtt{w}^{-1}(i+1) \in \{1,\dots,n-k\}$ and $\mathtt{w}^{-1}(i) \in \{n-k+1,\dots,k\}$, so we have to apply (3.4) to find the image $(\sigma_i\beta)(\mathbf{x}_\lambda)$ of $\mathbf{x}_\lambda$ under the braid action of $\sigma_i\beta$. In Algorithm 3.11, we have a initial strand for f that is labeled by i; thus for the new permutation $s_i f$, the same strand will now be labeled by $i+1$. In particular, we see that the algorithm applied to $s_i f$ will be identical to that of f, except in the case when the new label in f is $i+1$. In the case of the new permutation $s_i f$, this new label will be i, and will not include the last crossing of the corresponding interval braid for f. This matches (3.4) and finishes the proof. □

Example 3.18. *Let us choose $(k,n) = (4,6)$ and the Young diagram $\lambda = (2,2,2,2)$. Then the permutation $\mathbf{x}_\lambda$ reads*

$$\mathbf{x}_\lambda = \underbrace{(1,3)}_{5}\underbrace{(1,3)}_{6}\underbrace{(1,3)}_{1}\underbrace{(1,2)}_{2}\underbrace{(1,1)}_{3}\underbrace{(1,0)}_{4}$$

Let us choose the permutation $u = \sigma_2\sigma_4\sigma_3 \in S_6$. Then we have

$$\beta(u^{-1})\mathbf{x}_\lambda = \underbrace{(1,1)}_{3}\underbrace{(1,3)}_{6}\underbrace{(1,3)}_{1}\underbrace{(2,2)}_{2}\underbrace{(1,1)}_{4}\underbrace{(1,0)}_{5}$$

so that resulting braid reads

$$(\sigma_1)(\sigma_2)(\sigma_3\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)(\sigma_1).$$

If we instead choose the permutation $u_1 = \sigma_3\sigma_4\sigma_2$, then

$$\beta(u_1^{-1})\mathbf{x}_\lambda = \underbrace{(1,2)}_{4}\underbrace{(1,3)}_{6}\underbrace{(1,3)}_{1}\underbrace{(3,2)}_{2}\underbrace{(1,1)}_{3}\underbrace{(1,0)}_{5}$$

and thus the braid reads $(\sigma_1)(\sigma_3\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)(\sigma_2\sigma_1)$.

Finally, the choice of permutation $u_2 = \sigma_4\sigma_3\sigma_2$ leads to

$$\beta(u_2^{-1})\mathbf{x}_\lambda = \underbrace{(1,0)}_{2}\underbrace{(1,3)}_{6}\underbrace{(1,3)}_{1}\underbrace{(1,2)}_{3}\underbrace{(1,1)}_{4}\underbrace{(1,0)}_{5}$$

and the braid word $\sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)$. □

Let us conclude this section on the juggling braids $J_k(f) = J_k(u,w)$ by noticing that these braids are always a product of certain interval braids

$$J_k(u,w) = \sigma_{[a_n,b_n]}\cdots\sigma_{[a_1,b_1]}.$$

If we denote

$$J_k^{(1)}(u,w) = \sigma_{[a_{n-k},b_{n-k}]}\cdots\sigma_{[a_1,b_1]}, \quad J_k^{(2)}(u,w) = \sigma_{[a_n,b_n]}\cdots\sigma_{[a_{n-k+1},b_{n-k+1}]},$$

so that we can write $J_k(u,w) = J_k^{(2)}J_k^{(1)}$, and define

$$\mathcal{J}_k(u,w) := J_k^{(1)}(u,w)\Delta_k^{-1}J_k^{(2)}(u,w),$$

we obtain that $\mathcal{J}_k(u,w)$ is conjugate to $J_k(u,w)\Delta_k^{-1}$. Indeed

$$\mathcal{J}_k(u,w) = (J_k^{(2)}(u,w)^{-1})J_k(u,w)\Delta_k^{-1}(J_k^{(2)}(u,w)).$$

This equivalent expression $\mathcal{J}_k(u,w)$ for $J_k(u,w)$ will be used in Subsection 3.5.

Remark 3.19. The braid $J_k^{(2)} := \sigma_{[a_n,b_n]} \cdots \sigma_{[a_{n-k+1},b_{n-k+1}]}$ that appears in the decomposition of J_k is a positive lift of the minimal-length permutation that sorts $f(i_1), \dots, f(i_n)$ in decreasing order; so that $f(i_1) < \cdots < f(i_k)$ implies $J_k^{(2)} = \Delta_k$. For example, in Example 3.9 (i) we get $J_k^{(2)}(f) = \sigma_1$. □

This concludes our initial results and discussion on the juggling braids $J_k(f)$. In order to compare the juggling braids $J_k(f)$ to the Richardson braids $R_n(u_f, w_f)$, we introduce a third braid, the Le braid, which is the subject of the following subsection.

3.4. **Le Braid.** Let us define the Le braid $D_k(\daleth)$ associated to a Le-diagram $\daleth$. For that, let $\daleth$ be a Le-diagram and c a column of $\daleth$. Denote by $\delta(c) \subseteq [1,k]$ the set of indices for the rows in the column c that contain a dot, where the bottom row carries the number 1, and the index increases as we go up by height. The Le braid $D_k(\daleth)$ will be constructed by concatenating k-stranded tangles of the following type:

Definition 3.20 (Column Tangle)**.** Let c be a column in a Le-diagram with $\delta(c) \subseteq [1,k]$ its set of dots. By definition, the column tangle $\tau(c)$ is the tangle whose braid diagram in $[0,1] \times \mathbb{R}$ is constructed using the following k strands:

(i) $(k - |\delta(c)|)$ horizontal strands of the form $[0,1] \times \{i\}$, for $i \in [1,k] \setminus \delta(c)$,

(ii) $(|\delta(c)| - 1)$ strands given as the unique straight segments S_j, $j \in \delta(c) \setminus \{\min(\delta(c))\}$, where S_j unites the points $\{0\} \times \{j\}$ and $\{1, \mathfrak{m}(j)\}$, with $\mathfrak{m}(j) := \max\{k \in \delta(c) : k < j\}$,

(iii) A strand given as the unique straight segment S_0 uniting the points $\{0\} \times \{\min(\delta(c))\}$ and $\{1, \max(\delta(c))\}$,

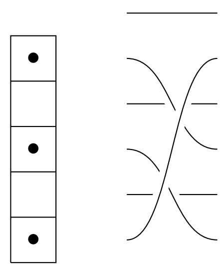
and resolving the crossings such that the strand S_0 of type (iii) is above all strands S_j of type (ii) and type (i), and all strands S_j of type (ii) lie above the stands of type (i). □

For instance, the tangle associated to a column with no dots is given by the trivial braid word, whereas the tangle associated to a column of height k with a dot in every entry is given by the braid word $\sigma_{[1,k]}^{-1}$.

Remark 3.21. Intuitively, the tangle $\tau(c)$ associated to a column c of $\daleth$ is drawn as follows. First, add empty boxes to the top of the column c until it acquires height k, and draw two parallel copies of this extended column at the same height. Then draw straight horizontal segments connecting the empty boxes of these two columns at the same level. After, draw strands that connect a box with a dot (on the left) with the box with a dot right below it (on the right), all above the horizontal strands. Finally, add a strand that connects the bottom box with a dot (on the left) with the top box with a dot (on the right), which runs above all previous strands. □

Definition 3.22 (Le braid)**.** The Le braid $D_k(\daleth)$ associated to the Le-diagram $\daleth$ is the tangle whose braid diagram is constructed by horizontally concatenating the column tangles of $\daleth$, where the tangle associated to the right-most column is drawn at the left, and the subsequent tangles are concatenated by being added to the right, in left to right order, according to the reverse order (right to left) of the corresponding columns in $\daleth$. □

Example 3.23. *Let us choose $k = 6$ and consider a column c of height 5 with $\delta(c) = \{1,3,5\} \subseteq [1,6]$. This column c and its associated tangle $\tau(c)$ read:*



Notice that $k = 6$, and thus we have added an additional trivial (horizontal) strand at the top, as the height of c was only five. The resulting tangle is 6-stranded. □

Our argument for Theorem 1.3.(i) uses an inductive construction of Le-diagrams. The upcoming Lemma 3.24 provides a simple characterization of Le-diagrams, of an inductive nature, which we will employ. A piece of terminology: a collection of dots in a column of a Le-diagram is said to be *top-adjusted* if for any dot in that collection, all boxes above it are filled with dots.

Lemma 3.24. *Let λ be a Young diagram and $\daleth$ a collection of dots in λ. The following rules hold:*

- *Suppose $\daleth$ has a column with no dots, and let $\daleth'$ be the result of removing this column, then $\daleth$ is a Le-diagram if and only if $\daleth'$ is.*

- *Suppose $\daleth$ has a row with no dots and let $\daleth'$ be the result of removing this row, then $\daleth$ is a Le-diagram if and only if $\daleth'$ is.*

- *Suppose $\daleth$ has no empty column nor empty row, let $\daleth'$ be the result of removing the last column from $\daleth$. Then $\daleth$ is a Le-diagram if an only if the last column in $\daleth$ is top-adjusted and $\daleth'$ is a Le-diagram.*

Proof. Parts (a) and (b) are immediate. For part (c), observe that if $\daleth$ has no empty row then each box in the last column of $\daleth$ is intersected by a horizonal line. If there is a dot in the last column, the northward line from it intersects all horizontal lines above it, and the intersections should contain dots. Therefore the last column is top-adjusted. □

Let us conclude this subsection by interpreting the above conditions on a Le-diagram $\daleth = \daleth(u, v)$ associated to a positroid pair (u, v) in terms of the affine permutation $f = u^{-1}t_k w$, as follows.

Lemma 3.25. *Let $\daleth = \daleth(u, v)$ be the Le-diagram associated to a positroid pair (u, v), and consider the associated affine permutation $f = u^{-1}t_k w$. The following hold:*

(a) If $\daleth(u, w)$ has an empty column then $f(i) = i$ for some i.
(b) If $\daleth(u, w)$ has an empty row then $f(i) = n + i$ for some i.

Proof. For Part (a), if $\daleth(u, w)$ has an empty column then $k < w(i) = u(i)$ for some $i \in [1, n]$. Hence we obtain

$$f(i) = u^{-1}t_k w(i) = u^{-1}w(i) = i.$$

For Part (b), if $\daleth(u, w)$ has an empty row, then $w(i) = u(i) < k$ for some $i \in \{1, \dots, n\}$. Thus we have

$$f(i) = u^{-1}t_k w(i) = u^{-1}(w(i) + n) = u^{-1}(w(i)) + n = i + n.$$

□

3.5. **Proof of Theorem 1.3.(i).** Let us prove Theorem 1.3.(i) by showing that both the Richardson braid $R_n(u, w)$ and the juggling braid $J_k(u, w)\Delta_k^{-1}$ are equivalent to the Le braid $D_k(\daleth(u, w))$. First, we start with the comparison between $R_n(u, w)$ and $D_k(\daleth(u, w))$, which essentially follows from our Proposition 3.5, and then proceed with the comparison between $J_k(u, w)\Delta_k^{-1}$ and $D_k(\daleth(u, w))$, which constitutes the majority of this subsection.

Let us denote $D_k(u, w) := D_k(\daleth(u, w))$ for the Le braid associated to the Le diagram $\daleth(u, w)$ corresponding to a positroid pair $u, w \in S_n$.

Theorem 3.26. *Let $u, w \in S_n$ be a positroid pair. Then, the Le braid $D_k(u, w)$ is equivalent to the Richardson braid $R_n(u, w)$, related by a sequence of braid moves and positive stabilizations.*

Proof. Consider the structure of the braid word for $R_n(u, w)$ produced in Proposition 3.5: the argument in its proof proceeded by iteratively sliding the words u_i^{-1}, composing $\beta(u)^{-1}$, $i \in [1, n-k]$. We can employ a similar procedure for analyzing the Le braid $D_k(u, w)$, by iteratively studying each column of the Le diagram $\daleth(u, v)$. Thus, it suffices to study the case of a single column $\daleth(u, v)$, i.e. $k = n - 1$, which corresponds to one iteration in the proof of Proposition 3.5, as $d = n - k = 1$.

Thus assume that the Young diagram for w consists of a single column, and denote its height by λ. Then the braid lifts of the positroid pair $u, w \in S_n$ read

$$\beta(w) = \sigma_k \cdots \sigma_{k-\lambda+1}, \quad \text{and} \quad \beta(u) = \sigma_{a_1} \cdots \sigma_{a_r},$$

for certain values a_j, $j \in [1, r]$, and $r \in \mathbb{N}$, such that $k - \lambda + 1 \leq a_r < a_{r-1} < \ldots < a_2, a_1 \leq k$, and the Richardson braid reads

$$R_n(u, w) = \beta(w)\beta(u)^{-1} = \sigma_k \cdots \sigma_{k-\lambda+1}\sigma_{a_r}^{-1} \cdots \sigma_{a_1}^{-1}.$$

Now, the dots in the Le diagram $⅃(u, w)$ then correspond to the set complement

$$\{k, \ldots, k - \lambda + 1\} \setminus \{a_1, \ldots, a_r\}.$$

In case $a_r = k - \lambda + 1$, we can perform a Reidemeister II move on the above braid word $R_n(u, w)$ which cancels the two generators $\sigma_{k-\lambda+1}\sigma_{a_r}^{-1}$. In parallel, on the Le-diagram $⅃(u, w)$, the top box is empty in this case and corresponds to a straight line; thus the braids coincide at this piece. Hence we proceed with the case $k - \lambda + 1 < a_r$. Then, we can slide the $\sigma_{a_j}^{-1}$ generators to the left and obtain the expression

$$R_n(u, w) = \sigma_{a_r-1}^{-1} \cdots \sigma_{a_1-1}^{-1}\sigma_k \cdots \sigma_{k-\lambda+1}.$$

By applying a positive Markov destabilization to the crossing σ_k, we arrive at the expression

$$\sigma_{a_r-1}^{-1} \cdots \sigma_{a_1-1}^{-1}\sigma_{k-1} \cdots \sigma_{k-\lambda+1}.$$

By direct inspection, this agrees with the column tangle in Definition 3.20, for the unique column we are studying. For instance, the case of a single dot in the column precisely corresponds to the cancellation

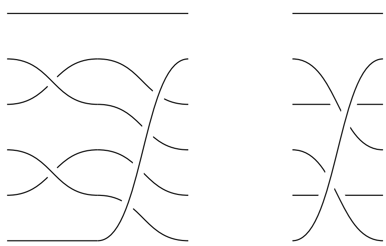
$$\sigma_k \cdots \sigma_{k-\lambda+1}\sigma_{k-\lambda_1+2}^{-1} \cdots \sigma_k^{-1} = \sigma_{k-\lambda+1}^{-1} \cdots \sigma_{k-1}^{-1}\sigma_k \cdots \sigma_{k-\lambda+1} \sim$$
$$\sigma_{k-\lambda+1}^{-1} \cdots \sigma_{k-1}^{-1}\sigma_{k-1} \cdots \sigma_{k-\lambda+1} = 1.$$

This shows the required equivalence between the Le braid $D_k(u, w)$ and the Richardson braid $R_n(u, w)$ for the case of a Le diagram with a single column; as explained above, the general case follows by applying this procedure iteratively. □

Example 3.27. *In Example 3.23, where we chose $k = 6$ and a column c of height 5 with dots $\delta(c) = \{1, 3, 5\} \subseteq [1, 6]$, the Richardson braid reads:*

$$R_n(u, w) = \sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_5^{-1} = \sigma_2^{-1}\sigma_4^{-1}\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2 \sim \sigma_2^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_2.$$

At the last step we apply a positive destabilization and remove σ_6. The result of this operation, which we now depict on the left, is equivalent to the column tangle $\tau(c)$ from Example 3.23, depicted on the right:



In both cases the sixth strand, on the top, interacts trivially with the other strands. □

Let us continue with the comparison between $J_k(u, w)\Delta_k^{-1}$ and $D_k(⅃(u, w))$. This is the main content of Theorem 3.31 below, whose proof contains several steps. In order to improve the clarity in the exposition, we first prove three lemmas, Lemma 3.28, 3.29 and 3.30, which will be used in proof of Theorem 3.31. These three lemmas address the changes of the braids $J_k(u, w)\Delta_k^{-1}$ and $D_k(⅃(u, w))$ upon certain simple modifications of the Le diagram $⅃(u, w)$, and thus the positroid pair $u, w \in S_n$.

Lemma 3.28. *Let $⅃(u, w)$ be a Le diagram with an empty column, $⅃' = ⅃(u', w')$ the Le diagram obtained by removing this empty column, where $u', w' \in S_{n-1}$ is the respective positroid pair. Then*

$$J_k(u, w) = J_k(u', w') \quad \text{and} \quad D_k(u, w) = D_k(u', w').$$

Proof. By Lemma 3.25, the associated k-bounded affine permutation has a fixed point and, by construction, the braid $J_k(u, w)$ does ignore the fixed points of f. Hence we obtain $J_k(u, w) = J_k(u', w')$. Now, an empty column in $⅃(u, w)$ yields a trivial column tangle according to 3.20, and thus $D_k(u, w) = D_k(u', w')$ as well. □

Lemma 3.29. *Let $\text{Ⅎ}(u,w) \subseteq \lambda$ be a Le diagram with no empty rows or columns, $\lambda^t_{n-k} - \mu$ the number of dots in its last column, and $\text{Ⅎ}' = \text{Ⅎ}(u',w')$ the Le diagram obtained by removing the last column, where $u', w' \in S_{n-1}$ is the associated positroid pair. Then*

$$D_k(u,w) = (\sigma_{k-1-\mu} \cdots \sigma_{k+1-\lambda^t_{n-k}}) D_k(u',w')$$

Lemma 3.29 is straightforward, i.e. the change of the Le braid $D_k(u,w)$ readily follows from Definition 3.22. Note that by Lemma 3.24 the last column in Ⅎ is top-adjusted. The analogous statement for the juggling braid $J_k(u,w)$ will now be proven in Lemma 3.30. For that, let us recall that we have a decomposition $J_k(u,w) = J^{(2)}(u,w)J^{(1)}(u,w)$ where, intuitively, $J^{(1)}(u,w)$ consists of the crossings appearing before n in the juggling diagram; see Subsection 3.3.

Lemma 3.30. *Let $\text{Ⅎ}(u,w) \subseteq \lambda$ be a Le diagram with no empty rows or columns, $\lambda^t_{n-k} - \mu$ the number of dots in its last column, and $\text{Ⅎ}' = \text{Ⅎ}(u',w')$ the Le diagram obtained by removing the last column, where $u', w' \in S_{n-1}$ is the associated positroid pair. Then*

$$J^{(1)}_k(u,w) = (\sigma_{k-1-\mu} \cdots \sigma_{k+1-\lambda^t_{n-k}}) J^{(1)}_k(u',w'), \quad \text{and} \quad J^{(2)}_k(u,w) = J^{(2)}_k(u',w').$$

Proof. Both $J_k(u,w)$ and $J_k(u',w')$ are products of interval braids, the precise form of which is obtained by Theorem 3.16. Let λ' be the Young diagram of the permutation $w' \in S_{n-1}$. Then Theorem 3.16 produces the two expressions

$$\mathbf{x}_\lambda := \underbrace{(k-\lambda^t_1+1,k-1)}_{k+1}\underbrace{(k-\lambda^t_2+1,k-1)}_{k+2}\cdots\underbrace{(k-\lambda^t_{n-k}+1,k-1)}_{n}\underbrace{(1,k-1)}_{1}\underbrace{(1,k-2)}_{2}\cdots\underbrace{(1,0)}_{k},$$

$$\mathbf{x}_{\lambda'} := \underbrace{(k-\lambda^t_1+1,k-1)}_{k+1}\underbrace{(k-\lambda^t_2+1,k-1)}_{k+2}\cdots\underbrace{(k-\lambda^t_{n-1-k}+1,k-1)}_{n-1}\underbrace{(1,k-1)}_{1}\underbrace{(1,k-2)}_{2}\cdots\underbrace{(1,0)}_{k}.$$

Hence, up to conjugation, the corresponding braids differ by the interval braid $\sigma_{k-1}\cdots\sigma_{k-\lambda^t_{n-k}+1}$. Since a reduced expression for the permutation u' can not include the crossing σ_{n-1}, the same is true for the braids associated to $\beta' \cdot (\mathbf{x}_{\lambda'})$ and $\beta' \cdot (\mathbf{x}_\lambda)$, where β' is a reduced lift of $(u')^{-1}$. Let us denote $\beta \cdot (\mathbf{x}_{\lambda'}) = (\vec{a}, \vec{b}, \mathtt{w})$.

Now, if β denotes a reduced lift of u^{-1}, we have that $\beta = \sigma_{n-\mu}\cdots\sigma_{n-1}\beta'$, which follows from the dots in the rightmost column of $\text{Ⅎ}(u,w)$ being top-aligned (which is the case by Lemma 3.24). Hence, we need to compute the element $(\sigma_{n-\mu}\cdots\sigma_{n-1}\beta')\cdot(\mathbf{x}_\lambda)$, i.e. the image of $\mathbf{x}_\lambda$ acted upon by the braid $(\sigma_{n-\mu}\cdots\sigma_{n-1}\beta')$. Applying σ_{n-1} to $\beta \cdot (\mathbf{x}_\lambda)$ will either produce the change

$$\underbrace{(k-\lambda^t_{n-k}+1,k-1)}_{n} \mapsto \underbrace{(k-\lambda^t_{n-k}+1,k-2)}_{n-1},$$

in the case $b_{n-1-k} < k-1$, by Lemma 3.17, or it will produce the change

$$\underbrace{(a_{n-1-k},k-1)}_{n-1}\underbrace{(k-\lambda^t_{n-k}+1,k-1)}_{n} \mapsto \underbrace{(k-\lambda^t_{n-k}+2,k-1)}_{n-1}\underbrace{(a_{n-1-k},k-1)}_{n}$$

for the critical case $b_{n-1-k} = k-1$, or equivalently $\mathtt{w}(n-1-k) = n-1$. In this case, the inequality $a_{n-1-k} < k - \lambda^t_{n-k} + 2$ holds and each one of $\sigma_{n-2}, \ldots, \sigma_{n-\mu}$ will either subtract 1 from the endpoint of the new interval, or add 1 to the initial point and move it to the left of the previous interval. Thus, at the end, we will have inserted an interval of the form $(k-\lambda^t_{n-k}+1+a, k-1-b)$ to $\beta\cdot(\mathbf{x}_{\lambda'})$, where $a+b=\mu$. By using a sequence of nested interval exchanges, we conclude that

$$J_k(u',w') = \sigma_{[a_{n-1},b_{n-1}]}\cdots\sigma_{[a_{n-k},b_{n-k}]}\sigma_{[a_{n-1-k},b_{n-1-k}]}\cdots\sigma_{[a_1,b_1]},$$
$$J_k(u,w) = \sigma_{[a_{n-1},b_{n-1}]}\cdots\sigma_{[a_{n-k},b_{n-k}]}\sigma_{[k+1-\lambda^t_{n-k},k-1-\mu]}\sigma_{[a_{n-1-k},b_{n-1-k}]}\cdots\sigma_{[a_1,b_1]}$$

from which the result follows. □

As a simple verification, note that the case $\mu = \lambda_{n-k}$ in Lemma 3.30 yields $J_k(u,w) = J_k(u',w')$, in accordance to Lemma 3.28 above. Finally, we can now show the equivalence between the juggling braid $J_k(u,w)\Delta^{-1}$ and the Le braid $D_k(u,w)$.

Theorem 3.31. *Let $u, w \in S_n$ be a positroid pair, and consider the juggling braid $J_k(u,w)\Delta^{-1}$, and the Le braid $D_k(u,w)$. Then*

$$D_k(u,w) = J^{(1)}(u,w)\Delta_k^{-1}J^{(2)}(u,w).$$

In particular, the juggling braid $J_k(u,w)\Delta^{-1}$ is conjugate to the diagram braid $D_k(u,w)$, and the Richardson braid $R_n(u,w)$ is related to them by a sequence of braid moves, conjugations and positive destabilizations.

Proof. Let us denote

$$\mathcal{J}_k(u,w) := J^{(1)}(u,w)\Delta_k^{-1}J^{(2)}(u,w),$$

to ease notation, and prove the theorem by induction on $n \in \mathbb{N}$. Note that for the empty $\daleth$-diagram both sides of the equation are the trivial braid, which gives us the base of induction. Now, given the positroid pair $u, w \in S_n$, we consider the associated Le diagram $\daleth(u,w)$. The induction step is now performed by modifying the Le diagram $\daleth(u,w)$ to a smaller Le diagram, in line with the inductive description if Lemma 3.24. There are three cases, as follows.

Case 1: There is an empty column in $\daleth(u,w)$. Let $\daleth' = \daleth(u',w')$ be the Le-diagram obtained by deleting this column. In this case, Lemma 3.28 implies that $D_k(u',w') = D_k(u,w)$ and $\mathcal{J}_k(u',w') = \mathcal{J}_k(u,w)$, and the induction step is immediate.

Case 2: There are no empty rows or columns in $\daleth(u,w)$. In this case, we let $\daleth' = \daleth(u',w')$ be the Le-diagram obtained by deleting the last column of $\daleth$. The induction hypothesis $\mathcal{J}_k(u',w') = D_k(u',w')$ and Lemmas 3.29 and 3.30 imply $\mathcal{J}(u,w) = D_k(u,w)$, as desired.

Case 3: There is an empty row in $\daleth(u,w)$. This case needs to be argued entirely, as none of the previous lemmas apply. Let $\daleth' = \daleth(u',w')$ be the result of removing this row, so that it has $k-1$ rows and $(n-k)$ columns, where we have denoted by $u', w' \in S_{n-1}$ the corresponding positroid pair.

First, assume for the moment that the empty row that we have removed is actually the bottom row of $\daleth$. Then $D_k(u,w) = D_{k-1}(u',w')$, where we abuse notation by considering $\mathrm{Br}_{k-1} \subseteq \mathrm{Br}_k$. On the juggling side, it is clear from the juggling diagram that we obtain the juggling braid $J_k(u,w)$ from $J_{k-1}(u',w')$ by inserting an arc going from n to $2n$; this implies that $J_k^{(1)}(u,w) = J_{k-1}^{(1)}(u',w')$ while $J^{(2)}(u,w) = (\sigma_1 \cdots \sigma_{k-1})J^{(2)}(u',w')$. Thus,

$$D_k(u,w) = D_{k-1}(u',w') = J_k^{(1)}(u',w')\Delta_{k-1}^{-1}J_{k-1}^{(2)}(u',w') = J_k^{(1)}(u,w)(\Delta_{k-1}^{-1}\sigma_{k-1}^{-1}\cdots\sigma_1^{-1})J_k^{(2)}(u,w).$$

and the result in this case follows from the equality $\Delta_k^{-1} = \Delta_{k-1}^{-1}\sigma_{k-1}^{-1}\cdots\sigma_1^{-1}$.

Let us now assume that we have removed the j-th row of $\daleth$, counting from top to bottom, which is an empty row. In particular, the j-th strand of $D_k(u,w)$ goes below all other strands. Let $\daleth_1 = \daleth(u_1,w_1)$ be the diagram obtained by removing the j-th row of $\daleth$ and attaching a new empty row at the bottom of $\daleth$, of maximal possible length $n-k$. In terms of partitions, we are replacing $\lambda = (\lambda_1, \ldots, \lambda_k)$ by the partition

$$(n-k, \lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_k),$$

where a hat means that an element is omitted. The Le braid $D_k(u_1,w_1)$ coincides with that of $D_k(u,w)$ after moving the j-th strand all the way to the bottom. This implies that

$$D_k(u_1,w_1) = (\sigma_{k-1}\cdots\sigma_j)D_k(u,w)(\sigma_{k-1}\cdots\sigma_j)^{-1}.$$

On the juggling side, let us first compare $J_k^{(1)}(u_1,w_1)$ with $J_k^{(1)}(u,w)$. In $J_k^{(1)}(u,w)$, we have a strand starting at the i-th position, say, that goes below all other strands and finishes in the j-th position, where $i \geq j$ and j is precisely the position of the row that we have removed from $\daleth$. The braid word $J_k^{(1)}(u_1,w_1)$ is obtained by pulling this strand all the way to the bottom. Thus, we have

$$J^{(1)}(u,w) = (\sigma_{k-1}\cdots\sigma_j)^{-1}J^{(1)}(u_1,w_1)(\sigma_{k-1}\cdots\sigma_i).$$

Still on the juggling side, let us now compare $J_k^{(2)}(u_1,w_1)$ with $J_k^{(2)}(u,w)$. The braid $J_k^{(2)}(u,w)$ has a strand starting at the j-th position and finishing at the $(k-i+1)$-st position, with i,j as before.

We then obtain $J_k^{(2)}(u_1,w_1)$ by pulling this strand so that now it starts at the bottom and ends at the top of the braid. Hence the $J^{(2)}$ piece changes according to

$$J^{(2)}(u,w) = (\sigma_1\cdots\sigma_{k-i})^{-1}J^{(2)}(u_1,w_1)(\sigma_{k-1}\cdots\sigma_j).$$

In conclusion, gathering the comparisons above, we obtain:

$$\begin{aligned} D_k(u,w) = & \ (\sigma_{k-1}\cdots\sigma_j)^{-1}D_k(u_1,w_1)(\sigma_{k-1}\cdots\sigma_j) \\ = & \ (\sigma_{k-1}\cdots\sigma_j)^{-1}J^{(1)}(u_1,w_1)\Delta_k^{-1}J^{(2)}(u_1,w_1)(\sigma_{k-1}\cdots\sigma_j) \\ = & \ J^{(1)}(u,w)(\sigma_{k-1}\cdots\sigma_i)^{-1}\Delta_k^{-1}(\sigma_1\cdots\sigma_{k-i})J^{(2)}(u,w) \\ = & \ J^{(1)}(u,w)\Delta_k^{-1}J^{(2)}(u,w), \end{aligned}$$

as required, where the second equality follows because $\daleth(u_1,w_1)$ has an empty row at the bottom. and the last equality follows from the identity $\sigma_\ell^{-1}\Delta_k^{-1} = \Delta_k^{-1}\sigma_{k-\ell}^{-1}$. □

3.6. **Matrix Braids and Proof of Theorem 1.3.(ii).** Let us finally introduce cyclic rank matrices $r=(r_{ij})$, their associated matrix braids $M_k(r)\in\mathcal{B}_k$, and conclude Theorem 1.3.(ii).

Definition 3.32. A cyclic rank matrix of type (k,n) is an array $r=(r_{ij})$ indexed by $(i,j)\in\mathbb{Z}^2$ satisfying the following conditions:

(i) $r_{ij}=0$ if $j<i$ and $r_{ij}=k$ if $i+n-1\le j$.
(ii) $r_{ij}-r_{(i+1)j}\in\{0,1\}$, $r_{ij}-r_{i(j-1)}\in\{0,1\}$, and $r_{ij}=r_{(i+n)(j+n)}$, for all $i,j\in\mathbb{Z}$.
(iii) If $r_{(i+1)(j-1)}=r_{(i+1)j}=r_{i(j-1)}$ then $r_{ij}=r_{(i+1)(j-1)}$.

Note that we can restrict to a grid $i\in[1,n]$ due to the condition $r_{ij}=r_{(i+n)(j+n)}$, for all $i,j\in\mathbb{Z}$. □

Given a cyclic rank matrix $r=(r_{ij})$, and each $i\in\mathbb{Z}$, there is a unique index $f(i)$ such that

$$r_{i\ f(i)} = r_{(i+1)\ f(i)} = r_{i\ (f(i)-1)} = r_{(i+1)\ (f(i)-1)}+1.$$

Then the map $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ defined by $f(i)=j$ if and only if

$$r_{ij}=r_{(i+1)j}=r_{i(j-1)}=r_{(i+1)(j-1)}+1,$$

defines a bounded affine permutation. In fact, this establishes a bijection between cyclic rank matrices and bounded affine permutations.

Definition 3.33. Let $r=(r_{ij})$ be a cyclic rank matrix of type (k,n), $(i,j)\in\mathbb{Z}^2$. By definition, the infinite matrix braid $M_k^\infty(r)\in\mathcal{B}_k$ is given by the tangle diagram obtained by drawn in $\mathbb{R}^2$ the five local tangles according to Figure 19.

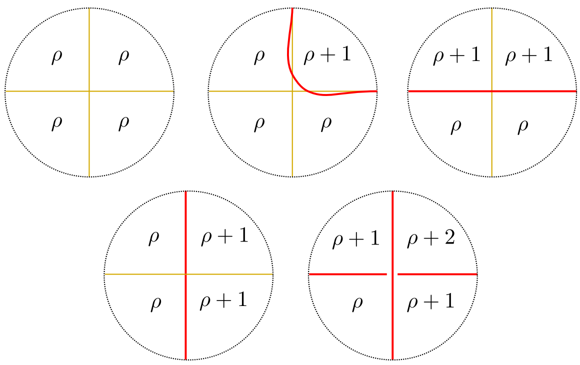


FIGURE 19. The local models for the matrix braid $M_k(r)$ associated to a cyclic rank matrix $r=(r_{ij})$, drawn near each four entries of the matrix. The value of a given entry r_{ij} is denoted by $r_{ij}=\rho$ and the braid is depicted in red strands. The yellow lines are used to separate the matrix entries of r.

By definition, the matrix braid $M_k(r)\in\mathcal{B}_k$ is obtained from the infinite matrix braid $M_k^\infty(r)\in\mathcal{B}_k$ by restricting its diagram to the grid $i\in[1,n]$. □

The conditions in Definition 3.32 imply that the matrix braid $M_k(r) \in \mathcal{B}_k$ is a k-stranded tangle. The following result concludes Theorem 1.3.(ii).

Theorem 3.34. *Let $r = (r_{ij})$ be a cyclic rank matrix of type (k,n), $(i,j) \in \mathbb{Z}^2$, and f its associated bounded affine permutation. The matrix braid $M_k(r) \in \mathcal{B}_k$ is equivalent to the braid $J_k(f)\Delta_k \in \mathcal{B}_k$.*

Proof. First, we note that

$$r_{i+1,j} - r_{i,j} = \begin{cases} 1 & \text{if } j < f(i), \\ 0 & \text{if } f(i) \leq j. \end{cases}$$

In particular, if $f(i) = i$ then $r_{ij} = r_{i+1,j}$ for all $j \geq i$, and there are no horizontal segments separating the rows i and $i+1$. In the other case, there is a horizontal line which starts at the bottom left corner of the square (i,i), makes a turn at the bottom left corner of the square $(i, f(i))$ and goes down after that. By examining the cases in Figure 19, we see that a vertical downward line cannot turn left and can turn right only at diagonal. If $f(i) \leq n$ then this vertical line will hit the diagonal, and corresponds to the arc connecting i with $f(i)$. In case $n < f(i)$, the vertical line would end at the bottom of the matrix braid $M_k(r) \in \mathcal{B}_k$, and thus connect to $f(i) - n$ after braid closure. □

3.7. **Reverse engineering.** Note that we can also try to reverse the logic of Subsection 3.3 and attempt to reverse engineer pairs of permutations $u, w \in S_n$ starting from a braid J_k so that $J_k = J_k(u,w)$. In this line, we pose the following question:

Question 3.35. *What is the set of positive k-stranded braids $\beta \in \mathrm{Br}_k$ that can be presented as $\beta = J_k(u,w)$, for some $w, u \in S_n$ and $n \in \mathbb{N}$?*

The general answer to this question seems complicated, e.g. see Theorem 3.16 for an algorithm computing the braid word for $J_k(u,w)$. Note that by Theorem 1.7, which will be proved in Section 4.2, such a braid $\beta = J_k(u,w)$ would have a non-empty braid variety $X(\beta)$, and thus a necessary condition for a braid β to be of this form is that it contains w_0 as a subword, cf. [11, Corollary 5.2]. The answer to the question becomes simpler in many special cases, for instance our computations above yield the following result:

Theorem 3.36. *Let $\gamma \in \mathrm{Br}_k$ be a k-stranded braid of the form*

$$\gamma = (\sigma_{k-1} \cdots \sigma_{k-\lambda_d}) \cdots (\sigma_{k-1} \cdots \sigma_{k-\lambda_1}), \quad \text{for some } \lambda_1 \geq \ldots \geq \lambda_d. \tag{3.5}$$

Then we have $\gamma\Delta_k = J_k(1,w)$, where $w \in S_{k+d}$ is the k-Grassmannian permutation corresponding to the partition λ. □

This result immediately implies the following:

Corollary 3.37. *Let $\gamma \in \mathrm{Br}_k$ be a k-stranded braid of the form*

$$\gamma = (\sigma_{k-1} \cdots \sigma_{k-\lambda_d}) \cdots (\sigma_{k-1} \cdots \sigma_{k-\lambda_1}), \quad \text{for some } \lambda_1 \geq \ldots \geq \lambda_d. \tag{3.6}$$

Then the braid variety $X(\gamma\Delta_k; w_{0,k})$ is isomorphic, up to a torus factor, to a positroid variety in $\mathrm{Gr}(k, k+d)$. □

Note that the presence of the torus factor in Corollary 3.37 is important. Indeed, the class of varieties Y, for which there exists a positroid Π in an appropriately big Grassmannian, such that Π is a direct product of Y with an algebraic torus, seems to be much larger that the class of positroids. This is what makes Question 3.35 interesting. In particular, we cannot always expect varieties $X(\gamma\Delta_k; w_{0,k})$ in Corollary 3.37 to be positroids. Note also that there are plenty of braids satisfying the condition (3.5). For example, let $\mathrm{FT}_i = (\sigma_{k-1} \cdots \sigma_{k-i+1})^i$ denote the full twist braid on the last i strands; then the braid

$$\gamma(a_2, \ldots, a_k; s) = \mathrm{FT}_2^{a_2} \cdots \mathrm{FT}_k^{a_k} (\sigma_{k-1} \cdots \sigma_1)^s$$

satisfies the condition (3.5) above for arbitrary $a_i, s \in \mathbb{N} \cup \{0\}$, $i \in [2,k]$. These braids play an important role in the articles [39, 34, 41]. Finally, we observe that the results of [32, 75] yield an explicit cluster structure on the braid variety $X(\gamma(a_2, \ldots, a_k; s)\Delta_k; w_{0,k})$; see also the discussion in Section 5.

4. Braid Varieties, Richardson varieties and Brick Manifolds

In this section, we succinctly argue that open Richardson varieties in Type A are braid varieties, which allows us to apply results of [62] to show that they satisfy the curious Lefschetz property, and use results of [53] to prove Theorem 1.7 in the introduction. In parallel, we also show that brick manifolds brick(β), for different braid words $\beta \in \mathcal{B}$ of a braid $[\beta] \in \mathrm{Br}_n$, provide different smooth projective good compactifications of the same affine braid variety $X(\beta)$, which only depends on $[\beta] \in \mathrm{Br}_n$.

4.1. **Braid varieties and open Richardson varieties.** Let us consider the flag variety $\mathscr{F}\ell_n = G/B$ for $G = \mathrm{GL}_n(\mathbb{C})$, B a Borel subgroup, and denote by $\mathscr{F}^A$ the flag associated to matrix $A \in \mathrm{GL}_n$. Namely, the i-th space in the flag $\mathscr{F}^A$ is spanned by the first i columns of the matrix A. Let us denote by $\mathscr{F}^{\mathrm{st}} = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \dots, e_{n-1} \rangle \subseteq \mathbb{C}^n)$ the standard flag.

Let $w \in W = S_n$ be a permutation, i.e. an element of the Weyl group of G. Then, the *Schubert cell* X°_w associated to $w \in S_n$ is defined to be

$$\overset{\circ}{X}_w := \{\mathscr{F} \in \mathscr{F}\ell_n \mid \dim(F^{\mathrm{st}}_p \cap F_q) = \#\{i \leq p \mid w(i) \leq q\} \text{ for all } p, q \in [1, n]\}.$$

The variety $\overset{\circ}{X}_w$ is actually an affine space of dimension $\ell(w)$, the length of w, which can be described as follows. Consider w as a permutation matrix and let E_{lm} be the (lm)-elementary matrix, so that the (s,t) entry of E_{lm} is the product $(\delta_{l,s})(\delta_{m,t})$ of Kronecker deltas. Inside the space of $(n \times n)$-matrices $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, consider the unique affine subspace M_w which contains $w \in M_n(\mathbb{C})$ and is spanned by all the matrices $E_{w(j),i}$ such that the pair (i,j) satisfies $(i,j) \in \mathrm{inv}(w)$.[10] It can be proven that the map

$$\iota : M_n(\mathbb{C}) \longrightarrow \mathscr{F}\ell_n, \quad A \mapsto \mathscr{F}^A,$$

restricts to an isomorphism between $M_{w^{-1}}$ and the Schubert cell $\overset{\circ}{X}_w$, i.e. $\iota(M_{w^{-1}}) \cong \overset{\circ}{X}_w$. Let us now relate to braid matrices. In fact, braid matrices serve as a parametrization of the affine spaces M_w, and thus the corresponding Schubert cells. This is the content of the following lemma, see e.g. [62, Section 5] and [11, Section 2].

Lemma 4.1. *Let $w \in S_n$ be a permutation and $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ a choice of reduced positive lift for w. Then the map*

$$\mathbb{C}^{\ell(w)} \to M_w, \qquad (z_1, \dots, z_\ell) \mapsto B^{-1}_{i_1}(z_1) \cdots B^{-1}_{i_\ell}(z_\ell)$$

is an isomorphism of affine algebraic varieties.

The story in the case of *opposite* Schubert cells is analogous. Briefly, consider the *anti-standard flag*

$$\mathscr{F}^{\mathrm{ant}} := (0 \subseteq \langle e_n \rangle \subseteq \langle e_n, e_{n-1} \rangle \subseteq \cdots \subseteq \langle e_n, \dots, e_2 \rangle \subseteq \mathbb{C}^n),$$

and let w_0 be the longest element of S_n, considered as a permutation matrix. The *opposite Schubert cell* is defined to be

$$\overset{\circ}{X}{}^w := \{\mathscr{F} \in \mathscr{F}\ell_n \mid \dim(F_p \cap F^{\mathrm{ant}}_q) = \#\{i \leq p \mid w_0 w(i) \leq q\} \text{ for all } p, q = 1, \dots, n\}$$

The variety $\overset{\circ}{X}{}^w$ is also an affine space, in this case of dimension $\ell(w_0 w) = \ell(w_0) - \ell(w)$. It can be described explicitly by considering

$$M^w := w w_0 + \sum_{(i,j) \in \mathrm{inv}(w)} \mathbb{C} E_{w(i), n-j},$$

the unique affine subspace M^w which contains $w w_0 \in M_n(\mathbb{C})$ and is spanned by all the matrices $E_{w(j), n-j}$ such that the pair (i,j) satisfies $(i,j) \in \mathrm{inv}(w)$. Similarly to above, the map $A \mapsto \mathscr{F}^A$ gives an isomorphism between the affine space $M^{w^{-1}}$ and the opposite Schubert cell $\overset{\circ}{X}{}^{w_0 w}$, and the analogue of Lemma 4.1 is the following

[10] Recall that an *inversion* of w is a pair (i,j) where $i < j$ and $w(i) > w(j)$, and $\mathrm{inv}(w)$ denotes the set of inversions of w; note that $\ell(w) = \#\,\mathrm{inv}(w)$.

Lemma 4.2. *Let $w \in S_n$ be a permutation and $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ a choice of reduced positive lift for w. Then the map*

$$\mathbb{C}^{\ell(w)} \to M^w, \qquad (z_1, \dots, z_\ell) \mapsto B_\beta(z) w_0$$

is an isomorphism of affine algebraic varieties. □

Lemmas 4.1 and 4.2 now suffice to show that if $\beta = \beta_1\beta_2$ with each one of β_1 and β_2 a reduced word, the braid variety are isomorphic to the intersection of a Schubert cell and an opposite Schubert cell.

Theorem 4.3. *Let $\beta_1, \beta_2 \in \mathcal{B}_n^+$ be two positive reduced braid words, and $w_1, w_2 \in S_n$ be their respective Coxeter projections. Then the map*

$$\iota : \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)} \longrightarrow \mathscr{F}\ell_n, \qquad (z_1, z_2) \mapsto \mathscr{F}^{B^{-1}_{\beta_1}(z_1)}, \quad (z_1, z_2) \in \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)},$$

restricts to an isomorphism

$$X(\beta_1\beta_2; w_0) \longrightarrow \iota(X(\beta_1\beta_2; w_0)) \cong \overset{\circ}{X}_{w_1} \cap \overset{\circ}{X}{}^{w_0 w_2^{-1}}$$

of affine algebraic varieties.

Proof. First, we verify that the image $\iota(X(\beta_1\beta_2; w_0))$ is indeed in the required intersection, i.e. that the flag $\mathscr{F}^{B^{-1}_{\beta_1}(z)}$ belongs to both Schubert cells $\overset{\circ}{X}_{w_1}$ and $\overset{\circ}{X}{}^{w_0 w_2^{-1}}$. By Lemma 4.1, the matrix $B^{-1}_{\beta_1}(z)$ belongs to the affine subspace $M_{w_1^{-1}}$, and thus $\mathscr{F}^{B^{-1}_{\beta_1}(z)} \in \overset{\circ}{X}_{w_1}$, as needed. For the inclusion $\mathscr{F}^{B^{-1}_{\beta_1}(z)} \in \overset{\circ}{X}{}^{w_0 w_2^{-1}}$, we note that we can write

$$B_{\beta_1}(z) B_{\beta_2}(z') w_0 = U$$

for some upper triangular matrix U. This implies that

$$B^{-1}_{\beta_1}(z) = B_{\beta_2}(z') w_0 U^{-1},$$

and, since U^{-1} is upper triangular, we conclude that $\mathscr{F}^{B^{-1}_{\beta_1}(z)} = \mathscr{F}^{B_{\beta_2}(z') w_0 U^{-1}} = \mathscr{F}^{B_{\beta_2}(z') w_0}$. Then Lemma 4.2 shows that the matrix $B_{\beta_2}(z') w_0$ belongs to M^{w_2}, and thus $\mathscr{F}^{B_{\beta_2}(z') w_0} \in \overset{\circ}{X}{}^{w_0 w_2^{-1}}$, as required. In order to show that ι restricts to a bijection, consider a flag $\mathscr{F} \in \overset{\circ}{X}_{w_1} \cap \overset{\circ}{X}{}^{w_0 w_2^{-1}}$. Thanks to Lemma 4.1 we can find a unique element $z_1 \in \mathbb{C}^{\ell_1}$ such that $\mathscr{F} = \mathscr{F}^{B^{-1}_{\beta_1}(z_1)}$. Now we claim that there exists a unique element $z_2 \in \mathbb{C}^{\ell_2}$ such that $(z_1, z_2) \in X(\beta_1\beta_2; w_0)$. Indeed, since $\mathscr{F} \in \overset{\circ}{X}{}^{w_0 w_2^{-1}}$, Lemma 4.2 implies that there exists a unique $z_2 \in \mathbb{C}^{\ell_2}$ such that $\mathscr{F} = \mathscr{F}^{B_{\beta_2}(z_2) w_0}$. The result is concluded. □

The intersections of Schubert cells considered above have a particular classical terminology. Namely, consider two permutations $u, w \in S_n$. Then the *open Richardson variety* associated to u, w is defined to be the intersection

$$\mathcal{R}^\circ(u, w) := \overset{\circ}{X}_w \cap \overset{\circ}{X}{}^u.$$

It can be verified that this intersection is nonempty if and only if $u \le w$ in Bruhat order; note that $\mathcal{R}^\circ(u, w)$ consists of a single point if $u = w$. In these terms, Theorem 4.3 states that all open Richardson varieties are braid varieties. We record this fact in the following

Corollary 4.4. *Let $u, w \in S_n$ be such that $u \le w$ in Bruhat order, and $\beta(w), \beta(u^{-1}w_0) \in \mathrm{Br}_n$ positive lifts of $w, u^{-1}w_0$. Then we have an isomorphism of affine algebraic varities*

$$X(\beta(w)\beta(u^{-1}w_0); w_0) \cong \mathcal{R}^\circ(u, w).$$

Remark 4.5. Note that we can use Corollary 4.4 together with Theorem 1.5 to give isomorphisms between Richardson varieties. For example, assume that u, w are permutations such that $u \le w$ in Bruhat order, and let s_i be a simple transposition satisfying $u < s_i u$ and $w < s_i w$ (resp. $u < u s_i$ and $w < w s_i$). Then, we have an isomorphism $\mathcal{R}^\circ(u, w) \cong \mathcal{R}^\circ(s_i u, s_i w)$ (resp. $\mathcal{R}^\circ(u, w) \cong \mathcal{R}^\circ(u s_i, w s_i)$). Indeed, it follows from Corollary 4.4 and Theorem 1.5(i) that we have $\mathcal{R}^\circ(u, w) \cong X(R_n(u, w)\Delta_n)/V$ and $\mathcal{R}^\circ(s_i u, s_i w) \cong X(R_n(s_i w, s_i u)\Delta_n)/V'$ (resp. $\mathcal{R}^\circ(u s_i, w s_i) \cong X(R_n(w s_i, u s_i)\Delta_n/V')$. But the braid words $R_n(u, w)\Delta_n$ and $R_n(s_i w, s_i u)\Delta_n$ are Δ-conjugate (resp. $R_n(u, w)\Delta_n$ and $R_n(w s_i, u s_i)\Delta_n$ are related by a Reidemeister II move) so we can use Theorem 1.5(i) again to obtain the desired isomorphism.

A. Mellit [62] used braid varieties to prove the *curious Lefschetz property*, defined by T. Hausel and F. Rodriguez-Villegas [46], for cohomology rings of character varieties. Namely, he stratified a vector bundle over any given character variety by vector bundles over braid varieties and proved the curious Lefschetz property for each braid variety by using further stratifications. Corollary 4.4 then immediately implies the following.

Corollary 4.6. *Open Richardson varieties for SL_n satisfy the curious Lefschetz property.*

This result was first conjectured by T. Lam and D. Speyer [56, Section 1.5.1], see also further discussion in a recent paper by P. Galashin and T. Lam [33].

4.2. **Proof of Theorem 1.7.** We are now in position to prove Theorem 1.7 from the introduction. Recall that a positroid pair (u,w) yields the open positroid variety $\Pi_{u,w} \subseteq \mathrm{Gr}(k,n)$ in the Grassmannian. Combining [53, Theorem 5.9] with Corollary 4.4, we obtain

$$\Pi_{u,w} \cong \mathcal{R}^\circ(u,w) \cong X(\beta(w)\beta(u^{-1}w_{0,n})),$$

where, as usual, $\beta(w)$, $\beta(u^{-1}w_{0,n})$ are positive braid lifts of their corresponding arguments.

Now, the Richardson braid $R_n(u,w)$ is $\beta(w)\beta(u)^{-1}$, which has negative crossings whenever $u \neq 1$. Nevertheless, since we can always find a reduced expression for $w_{0,n}$ with u as a prefix, the braid $R_n(u,w)\Delta_n$ is equivalent to the positive braid $\beta(w)\beta(u^{-1}w_{0,n})$. Thus, we are in position to apply our Theorem 1.5.(i) and obtain an expression for the positroid variety in terms of the braid variety $X(R_n(u,w)\Delta_n)$. Namely, we obtain an isomorphism of affine varieties

$$\Pi_{u,w} \cong X(R_n(u,w)\Delta_n)/V,$$

where V is a collection of vector fields on the variety $X(R_n(u,w)\Delta_n)$ that integrate to a free algebraic action of an additive group, as in Section 2. Moreover, since the passage from $R_n(u,w)$ to the Le-braid $D_k(\mathrm{J})$ involves only destabilizations but not stabilizations, we can use Theorem 1.5 (ii) together with Theorem 1.3(i) to, up to a trivial torus factor, express the positroid variety $\Pi_{u,w}$ in terms of the braid variety of a positive braid with k strands:

$$\Pi_{u,w} \cong X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^d$$

for some $d \in \mathbb{N} \cup \{0\}$, where $f = u^{-1}t_k w$ is the k-bounded affine permutation corresponding to the pair (u,w). It remains to show that $d = n-s-k$, where s is the number of fixed points of f in the interval $[1,n]$; this follows from a dimension count. Indeed, by Lemma 3.10 and [11, Corollary 5.22] we have

$$\dim\left(X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^d\right) = \ell(w) - \ell(u) - n + k + s + d.$$

In addition, see e.g. [53], we have that $\dim(\Pi_{u,w}) = \ell(w) - \ell(u)$ and thus $d = n-k-s$. □

4.3. **Braid varieties and brick manifolds.** Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathcal{B}_n^+$ be a positive braid word, and let us define the brick manifold $\mathrm{brick}(\beta)$ associated to β, following the work of L. Escobar [20]. These brick varieties $\mathrm{brick}(\beta)$ will provide natural smooth compactifications of our braid varieties $X(\beta)$.

By definition, the Bott-Samelson variety $\mathrm{BS}(\beta)$ associated to β is the moduli space of collections of flags $(\mathscr{F}^0,\dots,\mathscr{F}^\ell) \in \mathscr{F}\ell_n$ such that $\mathscr{F}^0$ is the standard flag and either $\mathscr{F}^j = \mathscr{F}^{j+1}$, or the two contiguous flags $\mathscr{F}^j, \mathscr{F}^{j+1}$ differ precisely in the i_{j+1}-subspace. This projective variety $\mathrm{BS}(\beta)$ contains a natural subvariety, which we called the open Bott-Samelson variety $\mathrm{OBS}(\beta)$ in [11], defined by the additional condition that two contiguous flags must be different, i.e. $\mathscr{F}^j \neq \mathscr{F}^{j+1}$ for every $j \in [0,\ell-1]$.

The Bott-Samelson variety $\mathrm{BS}(\beta)$ admits a natural projection map

$$m_\beta : \mathrm{BS}(\beta) \longrightarrow \mathscr{F}\ell_n, \quad m_\beta(\mathscr{F}^0,\dots,\mathscr{F}^\ell) := \mathscr{F}^\ell,$$

onto the last, rightmost, flag.

Definition 4.7. Let $\beta \in \mathcal{B}_n^+$ be a positive braid word. By definition, the *brick variety* $\mathrm{brick}(\beta)$ associated to β is

$$\mathrm{brick}(\beta) := m_\beta^{-1}(\delta(\beta)\mathscr{F}^{\mathrm{st}}),$$

where $\delta(\beta) \in S_n$ denotes the Demazure product of β. The associated *open brick variety* is defined as

$$\mathrm{brick}^\circ(\beta) := m_\beta^{-1}(\delta(\beta)\mathscr{F}^{\mathrm{st}}) \cap \mathrm{OBS}(\beta).$$

□

As a reminder, the Demazure product $\delta(\beta) \in S_n$ is the (unique) maximal permutation with respect to the Bruhat order such that β contains its positive braid lift. It is important to note that the brick manifold $\mathrm{brick}(\beta)$, unlike the braid variety $X(\beta)$, significantly depends on the braid *word* $\beta \in \mathcal{B}$, and not only on the braid $[\beta] \in \mathrm{Br}$.

Given $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$, denote its opposite braid word by $\reflectbox{$\beta$} := \sigma_{i_\ell} \cdots \sigma_{i_1}$. Braid varieties directly relate, up to this mirroring, to brick varieties, as follows:

Theorem 4.8. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ be a positive braid word, and consider the truncations $\beta_j := \sigma_{i_1} \cdots \sigma_{i_j}$, $j \in [1, \ell]$. The following holds:*

(*i*) *The algebraic map*

$$\Theta : \mathbb{C}^\ell \longrightarrow \mathscr{F}\ell_n^{\ell+1}, \quad (z_1, \ldots, z_\ell) \mapsto (\mathscr{F}^{st}, \mathscr{F}^1, \ldots, \mathscr{F}^\ell),$$

where $\mathscr{F}^j$ is the flag associated to the matrix $B^{-1}_{\reflectbox{$\beta$}_j}(z_{\ell-j+1}, \ldots, z_\ell)$, restricts to an isomorphism

$$\Theta : X(\reflectbox{β}; \delta(\beta)) \xrightarrow{\cong} \mathrm{brick}^\circ(\beta),$$

of affine varieties. In particular, the braid variety $X(\reflectbox{$\beta$}; \delta(\beta))$ is smooth.

(*ii*) *The complement to $X(\reflectbox{$\beta$}; \delta(\beta))$ in $\mathrm{brick}(\beta)$ is a normal crossing divisor. Its components correspond to all possible ways to remove a letter from β while preserving its Demazure product.*

Proof. For Part (i), we first verify $\Theta(X(\reflectbox{$\beta$}; \delta(\beta))) \subseteq \mathrm{brick}^\circ(\beta)$. For that, note that

$$B^{-1}_{\reflectbox{β}_{j+1}}(z_{\ell-j}, \ldots, z_\ell) = B^{-1}_{\reflectbox{β}_j}(z_{\ell-j+1}, \ldots, z_\ell) B^{-1}_{i_{j+1}}(z_{\ell-j}),$$

and thus the two flags $\mathscr{F}^j$ and $\mathscr{F}^{j+1}$ are indeed in position i_{j+1}, as required. In order to check that

$$\mathscr{F}^{B^{-1}_{\reflectbox{β}}(z_1, \ldots, z_\ell)} = \delta(\beta) \mathscr{F}^{\mathrm{st}},$$

we observe that we have $B_{\reflectbox{$\beta$}}(z_1, \ldots, z_\ell)\delta(\beta) = U$, where U is an upper triangular matrix, and hence $B^{-1}_{\reflectbox{$\beta$}}(z_1, \ldots, z_\ell) = \delta(\beta) U^{-1}$; from which this conclusion follows. The fact that the map Θ restricts to an isomorphism follows from Lemma 4.9 below, which is well-known. The smoothness claim follows immediately from [20, Theorem 3.3].

For Part (ii), we proceed as follows. For a subset $I \subseteq [1, \ell]$, let $\mathrm{brick}(\beta)^\circ_I \subseteq \mathrm{brick}(\beta)$ be defined by the conditions that $\mathscr{F}^{i-1} \neq \mathscr{F}^i$ if and only if $i \in I$, where $\mathscr{F}^0 = \mathscr{F}^{\mathrm{st}}$. For example, $\mathrm{brick}(\beta)^\circ_{[1,\ell]} = \mathrm{brick}(\beta)^\circ$. Now let $\mathrm{brick}(\beta)_I := \overline{\mathrm{brick}(\beta)^\circ_I}$, which is similarly defined by the condition that $\mathscr{F}^{i-1} = \mathscr{F}^i$ if $i \notin I$. Note that $\mathrm{brick}(\beta)_I \subseteq \mathrm{brick}(\beta)_J$ if $I \subseteq J$, and that $\mathrm{brick}(\beta)^\circ_I$ is nonempty if and only if $\delta(\beta_I) = \delta(\beta)$, where β_I is the subword of β indexed by I. Moreover, in this case we have natural isomorphisms

$$\mathrm{brick}(\beta_I)^\circ \to \mathrm{brick}(\beta)^\circ_I, \qquad \mathrm{brick}(\beta_I) \to \mathrm{brick}(\beta)_I.$$

Now, it is clear that we have

$$\mathrm{brick}(\beta) = \mathrm{brick}(\beta)^\circ \sqcup \bigcup_{I \subsetneq [1,\ell]} \mathrm{brick}(\beta)_I = X(\reflectbox{β}; \delta(\beta)) \sqcup \bigcup_{\substack{I \subsetneq [1,\ell] \\ \delta(\beta_I) = \delta(\beta)}} \mathrm{brick}(\beta_I).$$

If $\delta(\beta_I) = \delta(\beta)$, then $\mathrm{brick}(\beta_I)$ is a smooth variety of dimension $|I| - \ell(\delta(\beta))$, [20, Theorem 3.3]. Therefore, in this case

$$\mathrm{brick}(\beta_I) = \mathrm{brick}(\beta)_I = \bigcap_{j \notin I} \mathrm{brick}(\beta)_{[1,\ell] \setminus \{j\}} = \bigcap_{j \notin I} \mathrm{brick}(\beta_{[1,\ell] \setminus \{j\}}),$$

so $\mathrm{brick}(\beta_I)$ is a complete intersection, and the divisors $\mathrm{brick}(\beta_{[1,\ell] \setminus \{j\}}) \subseteq \mathrm{brick}(\beta)$ intersect transversally, if the intersection is nonempty. $\square$

The proof of the following lemma is straightforward.

Lemma 4.9. *Let us consider an invertible matrix $A \in \mathrm{GL}_n$ and $i \in [1, n-1]$. Then, the map*

$$\mathbb{C} \longrightarrow \mathscr{F}\ell, \quad z \longmapsto \mathscr{F}^{AB_i^{-1}(z)},$$

yields an isomorphism from $\mathbb{C}$ to the set of all flags that are in position i with respect to $\mathscr{F}^A$.

Remark 4.10. In fact, Corollary 4.4 follows from Proposition 4.8 by results of [20], since an open Richardson variety is a special case of an open brick variety (for the word considered in Corollary 4.4). Such resolutions of Richardson varieties via fibers of the Bott-Samelson map first appeared in a work of M. Balan [5] who reformulated constructions of M. Brion [7]; the latter were also studied in detail in [54]. □

Since the brick manifold brick(β) depends on the braid word $\beta \in \mathcal{B}$, and the braid variety $X(\beta)$ does not, Theorem 4.8.(ii) allows us to construct many different compactifications for a given braid variety, see Remark 4.14.

Example 4.11. *Let us consider the equivalent braid words*

$$\beta_1 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \quad \beta_2 = \sigma_1\sigma_2\sigma_2\sigma_1\sigma_2.$$

In both cases, the braid varieties are algebraic tori

$$X(\beta_1; w_0) \cong X(\beta_2; w_0) \cong (\mathbb{C}^*)^2.$$

The variety brick(β_1) *has* $X(\beta_1; w_0)$ *as an open stratum, and additional five codimension-1 strata – isomorphic to* $\mathbb{C}^*$ *– and five more codimension-2 strata, which are points. In fact,* brick(β_1) *is a toric degree 5 del Pezzo surface, i.e. the toric variety associated to the pentagon, and these various strata correspond to toric orbits. In contrast,* $X(\sigma_1\sigma_2^3; w_0)$ *is empty, so there can only be four codimension-1 strata – see below – and four codimension-2 strata. In fact,* brick$(\beta_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$*, which is a different toric variety.* □

Theorem 4.8.(ii) also allows us to compute the (equivariant) homology of braid varieties. Note that the equivariant cohomology of a (closed) Bott-Samelson variety is given by the Bott-Samelson bimodule. The brick manifold is the intersection of the Bott-Samelson variety with the graph of $\delta(\beta)$, and its equivariant cohomology can be computed similarly to [79].

Now, by Theorem 4.8.(ii), we can express the weight filtration on homology of the braid variety $X(\beta; w_0)$ in terms of homologies of brick manifolds brick(β_I) as follows. Consider the big complex

$$C_\bullet = \bigoplus_I H_*(\mathrm{brick}(\beta_I))$$

where the summation is over the subsets I such that $\delta(\beta_I) = \delta(\beta)$, and the differential is given by inclusions brick$(\beta_I) \hookrightarrow$ brick(β_J) for $I \subset J, |J| = |I|+1$. The homology of this complex is the E_2-page of the spectral sequence computing the homology of $X(\beta; w_0)$. Since all brick manifolds are smooth and projective, the weight filtrations in their homology agree with the homological gradings. Since higher differentials preserve weights [16, 17], the spectral sequence collapses at the E_2 page and we obtain

$$H_*(C_\bullet) = \mathrm{gr}^W H_*(X(\beta; w_0)).$$

Remark 4.12. By elaborating this argument further, one can compare the torus-equivariant homology $\mathrm{gr}^W H_T(X(\beta; w_0))$ to the Khovanov-Rozansky homology $\mathrm{HHH}^{a=n}(\beta\Delta)$ of top a-degree[11], see [51, 52]. By the main result of [40] (see also [6]) we have

$$\mathrm{HHH}^{a=n}(\beta\Delta) = \mathrm{HHH}^{a=0}(\beta\Delta^{-1}).$$

In particular, one can relate the torus-equivariant homology of the Richardson variety, corresponding to $\beta = R_n(u, w)\Delta$, to the Khovanov-Rozansky homology of $R_n(u, w)$ of bottom a-degree, in agreement with [33]. □

Finally, the open brick variety brick$^\circ(\beta)$ is the higher-dimensional stratum in the stratification of the brick variety brick(β) given in [20, Theorem 24], and we claim that all other strata of these stratification can also be realized as braid varieties, as follows. Let β' be a subword of β such that the Demazure product of β' coincides with that of β. Then, $X(\beta'; \delta(\beta))$ is a strata of brick(β), given by the conditions that $\mathscr{F}^j = \mathscr{F}^{j+1}$ whenever $i_{j+1} \notin \beta'$. This stratification is dual to the *subword complex* $(\beta, \delta(\beta))$, as defined by Knutson and Miller [55]. Subword complexes are defined for arbitrary pairs (β, π), where π is an element of a finite Coxeter group and β is a word in simple generators; the latter can also be seen as a positive braid word in the corresponding braid group. Knutson and Miller [55] proved that a subword complex is a sphere (or *spherical*) if and only if $\delta(\beta) = \pi$, and a ball,

[11] We thank Anton Mellit and Minh-Tam Trinh for explaining this to us and sharing the results of [78].

otherwise. Thus, brick manifolds bijectively correspond to spherical subword complexes, and they are stratified by braid varieties, with the adjacency of strata described by the dual complexes.

By directly translating a result of C. Ceballos, J. P. Labbé, and C. Stump [14] and its proof from the setting of spherical subword complexes to that of (open) brick manifolds, we can show the following.

Proposition 4.13. (*cf.* [14, Theorem 3.7]) *Given a positive braid β, let μ be a reduced expression of $\delta(\beta)^{-1}w_0$. Then we have isomorphisms*

$$\mathrm{brick}(\beta) \cong \mathrm{brick}(\beta\mu);$$
$$X(\beta;\delta(\beta)) \cong X(\beta\mu);w_0),$$

compatible with the inclusions $X(\beta;\delta(\beta)) \hookrightarrow \mathrm{brick}(\beta)$, $X(\beta\mu;w_0) \hookrightarrow \mathrm{brick}(\beta\mu)$.

Remark 4.14. Three brief comments might be in order. First, Proposition 4.13 explains why we can restrict ourselves to braids with the Demazure product w_0 without loss of generality, at least when considering smooth braid varieties. Contractible subword complexes might be related to other fibers of the projection m_β. To our knowledge, the geometry of such fibers is not well-understood; even their dimensions are, in general, not known.

Second, assume that β and β' are related by a Reidemeister III move and w is a permutation. Then, the relation between the subword complexes for (β, w) and (β', w) is described in the manuscript [44]. For the corresponding brick varieties – assuming $w = \delta(\beta) = \delta(\beta')$ – we heuristically foresee that brick(β) and brick(β') are related by at most one blow-up, followed by at most one blow-down. We will hopefully address this in more generality in future work. In fact, the main result of [43] essentially implies that each brick variety of the form brick$(\mathbf{cw_0})$, for any reduced expression $\mathbf{c}$ of a Coxeter element c and any reduced expression $\mathbf{w_0}$ of w_0 in S_n, can be obtained by a sequence of blow-ups from $(\mathbb{P}^1)^{n-1}$, and by a sequence of blow-downs from the toric variety of the c-associahedron. The precise geometric statements and their interpretations via quiver representations and g-vector fans will appear in the forthcoming work of the third author, which will be an extended translation of parts of his Ph.D. thesis [42].

Third, Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to the natural action of $(\mathbb{C}^*)^{n-1}$. Escobar [20] proved that the image of a brick manifold brick(β) under the corresponding moment map is a *brick polytope* of β, as introduced by V. Pilaud and C. Stump [70]. Notably, brick(β) is a toric variety of this polytope with respect to this torus action if and only if the word β is *root independent*, in the sense of [70]. Pilaud and Stump proved that the brick polytope of a root independent word β realizes its spherical subword complex; this is not true for an arbitrary braid word β. □

Now, suppose that we have $w = \delta(\beta)$. In order to further stratify each stratum $X(\beta'; w)$, we can use the algebraic weaves introduced in [11]. In short, if β_0 is a reduced lift of w, it follows from [11] that any simplifying weave $\mathfrak{w} : \beta \longrightarrow \beta_0$ gives a stratum of the form $(\mathbb{C}^*)^a \times \mathbb{C}^b$, where a is the number of trivalent vertices and b is the number of cups in $\mathfrak{w}$. Note that such weaves are in correspondence with weaves between $\beta \longrightarrow \beta_0$, and such correspondence preserves the number of trivalent vertices and cups. If $\mathfrak{w} : \beta \longrightarrow \beta_0$ is a Demazure weave, as defined in [11], then it must have $\ell(\beta) - \ell(\beta_0)$ trivalent vertices, and we find that the dimension of the brick manifold is precisely $\ell(\beta) - \ell(\beta_0)$; this dimension result goes back to [20]. In addition, by induction on $\ell(\beta) - \ell(\beta_0)$, we can show that the complement to the toric chart in $X(\beta; w)$ given by this weave can itself be further stratified by weaves. Let us conclude this section with an example of such a stratification.

Example 4.15. *Let us choose $n = 2$ and $\beta = \sigma_1^3$. The braid variety $X(\sigma_1^3; s_1)$ is a smooth surface in $\mathbb{C}^3$ defined by the equation*

$$X(\sigma_1^3; s_1) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_3(1 + z_1z_2) = 0\}.$$

If $1 + z_1z_2 = 0$, we get $z_1 = 0$, and thus come to a contradiction. Therefore, $1 + z_1z_2 \neq 0$. Then $z_3 = -\frac{z_1}{1+z_1z_2}$, and so $X(\sigma_1^3; s_1)$ is isomorphic to the complement

$$Y = \{(z_1, z_2) \in \mathbb{C}^2 : 1 + z_1z_2 \neq 0\}$$

of a smooth hyperbola in $\mathbb{C}^2$. The braid variety $X(\sigma_1^2; s_1)$ is isomorphic to $\mathbb{C}^*$ and $X(\sigma_1; s_1)$ is a point. The corresponding brick manifold is brick$(\sigma_1^3) \cong \mathbb{P}^1 \times \mathbb{P}^1$*, and these different braid varieties stratify it as follows. Consider the homogeneous coordinates $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, and denote $[0 : 1]$ by 0 and*

$[1:0]$ by ∞. Note that the homogenized hyperbola $\bar{C}$ contains the two points $(0,\infty)$ and $(\infty,0)$. The stratification of the brick variety $\mathbb{P}^1\times\mathbb{P}^1$ given by the braid varieties has the following seven strata

- *One top-dimensional stratum Y, which is the complement of the smooth hyperbola in $\mathbb{C}^2$.*
- *Three one-dimensional strata, each isomorphic to $\mathbb{C}^*$. Two such strata are given by*
$$\{(x,y)\in\mathbb{P}^1\times\mathbb{P}^1 : x=\infty, \quad y\neq 0,\infty\}, \quad \{(x,y)\in\mathbb{P}^1\times\mathbb{P}^1 : x\neq 0,\infty, \quad y=\infty\},$$
and the third one is the affine hyperbola itself.
- *Three zero-dimensional strata, each isomorphic to a point. These three points are $(\infty,0)$, $(0,\infty)$ and (∞,∞).*

Finally, the braid variety $X(\sigma_1^3;s_1)\cong Y$ admits two stratifications obtained by using algebraic weaves, as defined in [11]. *Each of them is a decomposition of Y of the form $(\{0\}\times\mathbb{C})\sqcup(\mathbb{C}^*\times\mathbb{C}^*)$:*

$$\begin{aligned} Y &= \{(z_1,z_2)\in\mathbb{C}^2 : z_1=0\}\sqcup\{(z_1,z_2)\in\mathbb{C}^2 : z_1\neq 0, z_2\neq -\frac{1}{z_1}\} \\ &= \{(z_1,z_2)\in\mathbb{C}^2 : z_2=0\}\sqcup\{(z_1,z_2)\in\mathbb{C}^2 : z_2\neq 0, z_1\neq -\frac{1}{z_2}\}. \end{aligned}$$

The reader is invited to explore these stratifications for the general 2-stranded case $\beta=\sigma_1^\ell\in\mathcal{B}_2$. □

5. Final Remarks and Conjectures on cluster $\mathcal{A}$-structures

Let us conclude this article with a few comments and conjectures on cluster $\mathcal{A}$-structures, also known as cluster K_2-structures, on braid varieties. The reader is referred to [26, 27] and [24, 23] for the necessary preliminaries on cluster structures, and see also [25, 45, 56, 64, 65, 66, 72, 29].

We introduce the following notation: we let β be a positive braid word with Demazure product $\delta(\beta)=w_0$ and $\mathbf{w_0}(\beta)$ be the rightmost subword of β which is a (reduced and positive) braid lift of w_0. Let $\mathbf{jump}(\beta):=\beta\backslash\mathbf{w_0}(\beta)$ be the subword of β, given by all letters **not** appearing in $\mathbf{w_0}(\beta)$; the subword $\mathbf{jump}(\beta)$ will be referred to as *jump set* of β, in line with the article [58]. In the setting of subword complexes [69], V. Pilaud and C. Stump refer to such a subword as the positive greedy facet of the subword complex of (β,w_0).[12] Let us now state precisely our conjectural understanding of braid varieties in relation to cluster structures:

Conjecture 5.1. *Let $\eta\in\mathcal{B}_n$ be a braid word which is equivalent to a positive braid word $\eta_+\in\mathcal{B}_n^+$ by a sequence of Reidemeister II and III moves, and Δ-conjugations, and $\mathcal{A}(\eta\Delta)$ its associated DG-algebra.*

(i) *There exists a quiver $\widetilde{Q}$, with exchange matrix of full rank, such that $\widetilde{Q}$ has $(l(\eta)-\binom{n}{2}-(n-1))$ mutable vertices, $(n-1)$ frozen vertices and the 0-th cohomology $H^0(\mathcal{A}(\eta\Delta))$ is isomorphic to the upper cluster algebra $\mathcal{U}(\widetilde{Q})$ over $\mathbb{C}$.*

(ii) *The mutable part Q of the quiver $\widetilde{Q}$ has a* green-to-red *(or* reddening*) sequence.*

(iii) *Q satisfies the strong version of the Louise property, as considered in* [66, Remark 4.7]. *In particular, $\mathcal{U}(\widetilde{Q})$ is locally acyclic, Q has a unique non-degenerate potential up to weak equivalence, and $H^*(Spec(H^0(\mathcal{A}(\eta\Delta))))$ is of mixed Tate type and split over $\mathbb{Q}$.*

(iv) *The 2-form on $Spec(H^0(\mathcal{A}(\eta\Delta)))$ defined by A. Mellit* [62]*, which yields the curious Lefschetz property for this affine variety, is a Gekhtman-Shapiro–Vainshtein form compatible with this cluster structure.*

(v) *Each cluster chart in $H^0(\mathcal{A}(\eta\Delta))$ corresponds to an embedded exact Lagrangian filling of the Legendrian link $\Lambda(\eta\Delta)$ associated to the Lagrangian Pigtail closure of $\eta\Delta$.*

[12]Note that the subword $\mathbf{w_0}(\beta)$ generalizes the notion of the positive distinguished subexpression of v in a reduced expression of w, for a pair of permutations $u\leq w$, as introduced by B. Marsh and K. Rietsch [61], following V. Deodhar [18].

(vi) *The Lagrangian filling defined by the pinching sequence given opening the crossings in* $\mathbf{jump}(\eta_+)$ *from left to right is an initial cluster chart.*

(vii) *The exchange type of the mutable part of* Q *for such choice of the initial chart is preserved under Reidemeister II moves, Reidemeister III moves and* Δ*-conjugations of the braid word* η*. Each such move gives rise to a quasi-cluster transformation of Spec*$(H^0(\mathcal{A}(\eta\Delta)))$*. In particular, it preserves the set of cluster monomials and the totally positive part of the variety.*

(viii) *Each positive stabilization adds one frozen vertex to the quiver* $\widetilde{Q}$*, and each positive destabilization specializes one frozen variable to 1.*

In particular, $H^0(\mathcal{A}(\eta\Delta))$ *equals the cluster algebra* $\mathcal{A}(\widetilde{Q})$ *and admits a theta basis and a generic basis, both parametrized by tropical points of the corresponding cluster* $\mathcal{X}-$*variety.* □

In fact, for each subword of η_+ which is a braid lift of w_0, we may choose a pinching ordering for the crossings in the complement of such a word. We expect that any exact Lagrangian filling obtained in this manner yields a cluster seed in $H^0(\mathcal{A}(\eta\Delta))$. In general, this will not be a 1-to-1 correspondence: many pinching sequences often give rise to the same cluster chart. In [11], we described all toric charts appearing in such a way in terms of *Demazure weaves*, following [13], and gave a precise notion of equivalence and mutations of such weaves. We do expect that one can associate a quiver to each Demazure weave, and each mutation of such a quiver can be realized via a mutation of the weave. This is straightforward to check for 2-stranded braids, see [11]. J. Hughes [47] proved this statement for weaves for 3-stranded braids whose corresponding cluster algebras are of type D, and B. H. An, Y. Bae, and E. Lee [1] independently proved that the images under iterated Coxeter transformations of quivers of cluster type ADE can be realized via weaves for 3-stranded braids. Therefore, for such braids, all clusters correspond to certain Lagrangian fillings of the appropriate Legendrian links.

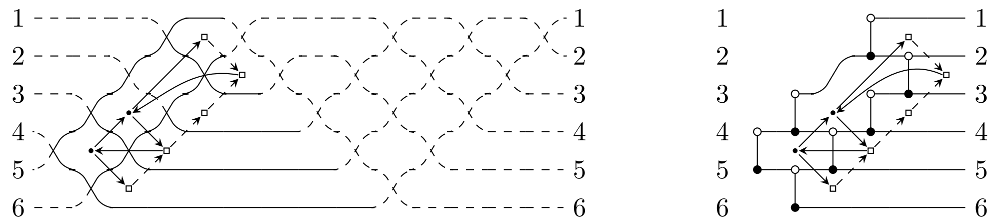
Remark 5.2. Despite its success so far, e.g. see [1, 11, 13, 47], the approach via weaves also presents certain challenges; mostly in the dissonance between algebraic and geometric intersection numbers. In [11], we adapted results of Y. Pan [68] for the case of 2-stranded braids and presented some calculations in a larger generality where this strategy can be successfully implemented. We hope to address the general case in future work. □

Let us focus on Conjecture 5.1, providing evidence for it and precisely setting up the larger context in which it lies. For certain classes of braids β as in its statement, there are known cluster $\mathcal{A}$-structures on their braid varieties. Let us succinctly describe known cases and the relations between them.

Positroid strata. For positroid varieties $\Pi_{u,w}$ – which, as we proved, are isomorphic to $H^0(\mathcal{A}(\eta\Delta))$, for $\eta = R_n(u,w)\Delta_n$ – Conjecture 5.1.(i) was proved by P. Galashin and T. Lam [32], based on works of J. Scott [74], G. Muller and D. Speyer [66], B. Leclerc [57], and K. Serhiyenko, M. Sherman-Bennett, and L. Williams [75]. Explicitly, they proved that $\widetilde{Q}$ can be taken as the quiver associated to the Le-diagram of (u,w). The same $\widetilde{Q}$ was associated by G. Muller and D. Speyer [66] to a *plabic graph* corresponding to (u,w), who further proved that Q satisfies the strong version of the Louise property. N. Ford and K. Serhiyenko [28] proved that such Q admit green-to-red sequences. To summarize, for $\eta = R_n(u,w)\Delta_n$ and $\eta_+ = \beta(w)\beta(u^{-1}w_0)$, parts (i), (ii), and (iii) of Conjecture 5.1 are known.

Note that one has cluster structures on coordinate rings both of a positroid variety and of its affine cone, and some of the references above actually deal with the latter. The former can be obtained from the latter by specializing one frozen variable to 1; see [67] for a detailed discussion. We expect part (vii) to be related to recent results of C. Fraser and M. Sherman-Bennett on quasi-cluster transformations of cluster structures on coordinate rings of (affine cones over) positroid varieties, formulated in terms of relabeled plabic graphs [30].

In terms of Le-diagrams, the Le-diagram associated to (u,w) corresponds to a choice of a pair $(\mathbf{v},\mathbf{w})$, where $\mathbf{w}$ is a certain reduced expression of w, which we can identify with an explicit braid lift $\beta(w)$, and $\mathbf{v}$ is the leftmost reduced expression of v in $\mathbf{w}$. The braid word $\beta(w)\beta(u^{-1}w_0)$, which is equivalent to the braid word $R_n(u,w)\Delta_n$, has the Demazure product w_0, and the complement to $\mathbf{v}$ in $\beta(w)$ is a facet of the subword complex of (β, w_0). Regarding the quivers and cluster coordinates, we have:



(A) $\beta(w)\beta(u^{-1}w_{0,6})$. Here $w_4 = (s_4s_3s_2s_1)(s_5s_4s_3s_2)$ is the maximal 4 - Grassmannian permutation in S_6; $u = s_2$; $\beta(u^{-1}w_{0,6}) = \sigma_1\sigma_2(\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1)$.

(B) The bridge diagram for $\beta(w)\beta(u^{-1}w_{0,6})$.

FIGURE 20. The wiring diagram for w_4 with the unique crossing in $u = s_2$ being removed and the bridge diagram associated to (u, w_4). The quiver defines a cluster structure on the coordinate ring of the open positroid $\Pi_{u,w} \in Gr(4,6)$.

- For such words, the quiver can be drawn by a procedure essentially described by R. Karpman [50], who related wiring diagrams to plabic graphs. In short, it goes as follows. We draw a wiring diagram for w (we list the crossing of $\beta(w)$ from left to right). We replace each crossing in the complement to $\mathbf{v}$ by a dimer (which we can safely understand just as a vertical edge, as in the brick diagram of $\beta(w)$). We then remove the tail of each strand to the left of the leftmost dimer. The result is called a *bridge diagram*. In the case of positroids, Karpman proved that such diagrams are planar. Thus, one has well-defined *bounded* and *unbounded* regions in the complement to strands and dimers. In each region bounded on the left by a dimer, we put a vertex of a quiver. Bounded regions correspond to mutable vertices and unbounded ones correspond to frozen vertices.

  Galashin and Lam [32] explained how to draw the arrows in such a diagram. In fact, they used a Le-diagram to draw the arrows. A notable difference in the construction is that they read the Le-diagrams in the opposite direction. In our terminology, this means that they take the *rightmost* reduced expression of v^{-1} in a reduced word for w^{-1}. The complement is then precisely the jump set in $\beta(w^{-1})\beta(uw_0)$. The bridge diagram has to be reflected across the vertical axis. The frozen vertices now correspond to the leftmost regions.

- The cluster variables are defined via an explicit parametrization of the initial toric chart of this positive distinguished subexpression, described by Marsh and Rietsch [61]. We expect that this construction might be interpreted in Floer theoretic terms of exact Lagrangian fillings, and such a translation would verify point (vi) of Conjecture 5.1 for the word $\beta(w^{-1})\beta(uw_0)$.

  Note that if we translate these results to our initial word $\beta(w)\beta(u^{-1}w_0)$, we would find that the subword for w_0 defining the initial cluster is neither rightmost, nor the leftmost. This means that in this setting, we can assign vertices of the quiver to the crossings of a certain non-greedy facet of a subword complex of $(\beta(w)\beta(u^{-1}w_0), w_0)$ and obtain a parametrization of a toric chart via translating the parametrization of Marsh and Rietsch [61]. An example of a such a quiver and the corresponding bridge diagram are given on Figure 20.(A) and (B), respectively. For the sake of completeness, we drew all the crossings of $\beta(u^{-1}w_0)$ to show which subword of $\beta(w)\beta(u^{-1}w_0)$ we use.

Open Richardson varieties. More generally, for open Richardson varieties, parts (i) and (iii) were conjectured by T. Lam and D. Speyer [56]. By now, there are at least two known constructions of upper cluster algebra structures on the coordinate rings $\mathbb{C}[\mathcal{R}^\circ(u,w)]$. B. Leclerc [57] proved that the upper cluster algebra (and thus the honest cluster algebra) corresponding to a certain cluster-tilting object in a certain category is a subalgebra in $\mathbb{C}[\mathcal{R}^\circ(u,w)]$. His approach actually covers open Richardson varieties in types ADE, but we will concentrate on type A. His algebra is obtained by localizing the image of the cluster character map at a certain explicit set of functions. Leclerc conjectured that this upper cluster algebra is, in fact, the entire $\mathbb{C}[\mathcal{R}^\circ(u,w)]$, and, moreover, it coincides with the corresponding cluster algebra. The main issue is that the cluster seed corresponding to this object is not defined explicitly. Nonetheless, Leclerc proved his conjecture in two cases: when the cluster algebra is of finite type and when w admits a factorization $w = vu$ with $l(w) = l(v)+l(u)$. In the second case, the varieties are called *skew Schubert varieties* in [75]. Recently, Ménard [63] found an explicit cluster seed in Leclerc's cluster algebra for a certain cluster-tilting object (which is conjectured to coincide with Leclerc's cluster-tilting object). Moreover, he related its quiver to the one of a cluster-tilting object considered by C. Geiss, B. Leclerc, and J. Schröer in [37] by an explicit sequence of mutations, followed by freezing of some vertices, followed by deletions of frozen vertices. The quiver of Geiss-Leclerc-Schröer is known to have a green-to-red sequence [37] (which is conjectured to be a maximal green sequence), and the property of having such a sequence is preserved under all these steps by results of G. Muller [65]. P. Cao and B. Keller [9] combined these results to deduce the existence of a green-to-red sequence for Ménard's quiver. They also announced a proof of Conjecture 5.1.(i) for Richardson varieties, based on this construction and on properties of cluster algebras whose mutable quivers admit green-to-red sequences. The positive distinguished subexpression for v in $\beta(w)$ plays a role in Ménard's construction; however, we do not have enough intuition yet to claim that this approach should prove Conjecture 5.1.(vi).

Independently, G. Ingermanson [48] gave another construction of an upper cluster algebra structure on $\mathbb{C}[\mathcal{R}^\circ(u,w)]$ (only in type A), thus giving another proof of Conjecture 5.1.(i) in this generality. Her construction uses bridge diagrams, which are again related to positive distinguished subexpression for v in $\beta(w)$, now for so-called *unipeak* reduced expressions $\beta(w)$. This can be seen as a direct generalization of the construction of Galashin and Lam [33]; thus, for this cluster structure, we expect Conjecture 5.1.(vi) to hold. It is an interesting open question whether the constructions of Leclerc-Ménard and of Ingermanson produce the same cluster $\mathcal{A}$-structure on $\mathcal{R}^\circ(u,w)$.

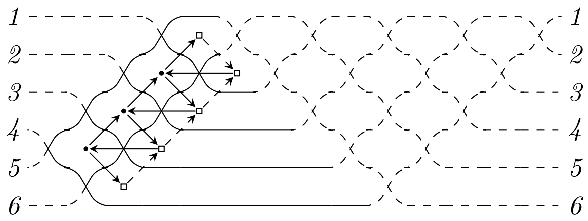
Note that bridge diagrams in this general Richardson case (even in type A) are not necessarily planar. Thus, the recipes of Karpman and Galashin-Lam to draw quivers cannot be applied directly. This explains the difficulties involved.

A few more cases. For a braid word η_+ of the form $\bar{\eta}_+\Delta$, parts (i), (ii) and (v) of Conjecture 5.1 were proved in [35, 76]. In this case, the jump set of $\bar{\eta}_+\Delta$ is $\bar{\eta}_+$, and Conjecture 5.1.(vi) also follows from their works. Note that in [35] part (i) was proved only over a field of characteristic 2, but it can be combined with [12] to define a cluster structure over $\mathbb{C}$. By rotating the corresponding wiring diagrams by 180° or by conjugating by Δ, we can translate results of [35] to the case of braids of the form $\Delta\bar{\eta}_+$. If we retain their parametrization of cluster charts, we obtain that parts (i), (ii) and (v) of Conjecture 5.1 are satisfied for such braids, when one takes the crossings in $\bar{\eta}_+$ as the vertices of the quiver. In other words, one takes the complement to the leftmost expression of w_0 in $\Delta\bar{\eta}_+$. Note that the pinching sequence of the filling corresponding to such a cluster consists of crossings in $\bar{\eta}_+$ opened *from right to left.*

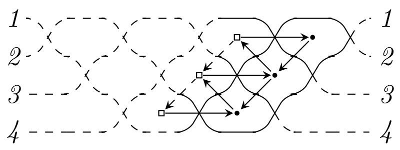
Finally, assume that $w = vu, l(w) = l(v) + l(u)$, and w is k-Grassmannian. Then the quiver corresponding to the Le-diagram can be constructed from the Young diagram for v by the same rule. Indeed, we can take positive lifts of v and u to get a positive lift of w, and then $\beta(w)\beta(u^{-1}w_0)$ equals $\beta(v)\Delta$ (or is equivalent to it by a sequence of Reidemeister III moves). Thus, one can take the wiring diagram of $\beta(v)$ and construct a quiver from it by the same rules as in [32]. This was done (in fact, before [32]) by K. Serhiyenko, M. Sherman-Bennett and L. Williams [75]. By comparing to [35], we see that the quiver is, in fact, the same in both constructions. In other words, the cluster structures on (open) skew Schubert varieties in Grassmannians can be interpreted in terms of the pinching combinatorics for exact Lagrangian fillings.

Example 5.3. *Let us give an example in the case of the big positroid cell in* $\mathrm{Gr}(k,n)$*. The Richardson braid is the shuffle braid* $\beta(w_k)$*, so its crossings form a rectangle* $k\times(n-k)$*. The quiver of* [32] *reflected across the vertical axis gives rise to a so-called* rectangular seed*, see* [75]*. In fact, this particular cluster seed was first described in* [38]*, and the wiring diagram and the bridge diagrams are just the usual diagrams for* w_k*; we draw the quiver as in* [35]*. The subword of* $\beta(w_k)\Delta$ *that we use is the jump set, and we open the crossings of* $\beta(w_k)$ *from left to right.*

We can also draw the corresponding juggling braid: in this case it is $\Delta_k(\sigma_{k-1}\ldots\sigma_1)^{(n-k)} \in \mathrm{Br}_k^+$*. Since it begins by* Δ_k*, the vertices of the natural quiver correspond to the complement to the leftmost subword for* Δ_k*, and we open the crossings of* $(\sigma_{k-1}\ldots\sigma_1)^{(n-k)}$ *from right to left. The diagrams and quivers for* $k=4, n=6$ *are drawn on Figure 21. Note that the quiver for* $J_4(f)$ *has* $4-1=3$ *frozen vertices, while the quiver for* $\beta(w_k)\Delta$ *has* $6-1=5$ *frozen vertices.* □



(A) $R_6(1, w_4)\Delta_6$ for the shuffle braid $R_6(1, w_4) = \beta(w_4) = (\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2) \in \mathrm{Br}_6^+$. Here w_4 is the maximal 4-Grassmannian permutation in S_6.



(B) $J_4(f)$ for the (4, 2) torus braid $(\sigma_3\sigma_2\sigma_1)^2 \in \mathrm{Br}_4^+$.

FIGURE 21. Wiring diagrams and quivers for the positive braid words $R_n(1, w_k)\Delta_n$ and $J_k(f)$ for $(k, n) = (4, 6)$. The corresponding DG-algebras are $\mathcal{A}(R_6(1, w_4)\Delta_6^2)$ and $\mathcal{A}(J_4(f)\Delta_4)$. The braid varieties are the big positroid cell in $\mathrm{Gr}(4, 6)$ and its quotient by $(\mathbb{C}^*)^2$, respectively. Each of them is a cluster $\mathcal{A}$-variety of type A_3.

Remark 5.4. P. Galashin and T. Lam [33] introduced the *positroid configuration space* as a quotient of the positroid whose braids close up to a knot by $(\mathbb{C}^*)^{n-1}$. Such varieties appear in the particle physics, see [2, 3]. In terms of cluster algebras, this quotient is obtained by specializing all frozen variables to 1. A special case is the Catalan variety, which is obtained by such a quotient from the big positroid cell in $\mathrm{Gr}(k, n)$ for $\gcd(k, n) = 1$. The variety $X(J_k(f))$ considered in the previous example sits between the big positroid cell and the Catalan variety: we specialize to 1 precisely $(n - k)$ out of $(n - 1)$ frozen variables. $\square$

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