ALGEBRAIC WEAVES AND BRAID VARIETIES

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Abstract. In this manuscript we study braid varieties, a class of affine algebraic varieties associated to positive braids. Several geometric constructions are presented, including certain torus actions on braid varieties and holomorphic symplectic structures on their respective quotients. We also develop a diagrammatic calculus for correspondences between braid varieties and use these correspondences to obtain interesting stratifications of braid varieties and their quotients. It is shown that the maximal charts of these stratifications are exponential Darboux charts for the holomorphic symplectic structures, and we relate these strata to exact Lagrangian fillings of Legendrian links.

1. Introduction

This article studies braid varieties, a class of affine algebraic varieties associated to positive braids, and their relation to contact and symplectic geometry. First, the geometric properties of braid varieties are studied, including the construction of torus actions and holomorphic symplectic structures on their quotients. Then, we construct correspondences between these braid varieties by using certain moduli spaces associated to weaves, a class of labeled planar diagrams. These geometric correspondences are shown to induce valuable stratifications for braid varieties and their quotients, also unifying known constructions of A. Mellit, in the case of character varieties, and M. Henry and D. Rutherford, in the case of augmentation varieties.

The diagrammatic calculus based on weaves, presented in Section 5, allows for direct and explicit computations, and we provide new constructions of embedded exact Lagrangian fillings for Legendrian links through combinatorial methods. The main results of the article are Theorems 1.1, 1.4, 1.8 and 1.10, and several detailed examples are provided throughout the manuscript. In particular, we believe that both the construction of a holomorphic symplectic structure on augmentation varieties, developed in Section 3, and the relation between weaves and cluster structures, as discussed in Section 6, is of value for contact and symplectic geometry.

1.1. **Context.**

Legendrian links in contact 3–manifolds [1, 4, 38] are central in contact and symplectic geometry. Legendrian fronts, immersed planar cuspidal curves, arise in topology, as Cerf diagrams [2, 10, 24], in differential equations, as Stokes data for irregular singularities [5, 86, 87], and in analysis, as wavefront sets [49, 50, 61]. In this article, we use that a positive braid β naturally gives rise to a Legendrian link $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ [16, 38].

Associated to a Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, there exist two geometrically defined moduli spaces: the moduli space of microlocal sheaves in $\mathbb{R}^2$ microlocally supported at Λ [44, 56, 57], and the moduli space of exact Lagrangian fillings $L \subseteq (\mathbb{R}^4, \omega_{\mathrm{st}})$, with boundary $\partial L = \Lambda$ [2, 38, 16]. Note that the latter can be understood as the (geometric part of the) moduli space of objects of the Fukaya category of $(\mathbb{R}^4, \omega_{\mathrm{st}})$ partially wrapped at Λ. For the Legendrian links $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, these moduli are algebraic stacks, often smooth algebraic varieties. The present manuscript studies a collection of algebraic varieties associated to a positive braid β, including and generalizing these two moduli spaces, and new correspondences between them. These algebraic correspondences are often induced by geometric exact Lagrangian cobordisms between Legendrian links, and can in general be described with a diagrammatic calculus, as we will show, building on the recent work of the first author with E. Zaslow [19].

In summary: we introduce the class of *braid varieties*, study torus actions and their quotients, construct correspondences and morphisms between them, and develop a diagrammatic calculus associated to these correspondences. As we establish these results, we prove several theorems of interest, including the fact that the augmentation variety associated to $\Lambda(\beta)$ admits a holomorphic symplectic structure, and explain the relation between A. Mellit's stratification of character varieties [68] and the ruling stratification of the augmentation variety [45, 46]. Note that holomorphic symplectic structures play

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a central role in the study of moduli spaces of connections [7, 8], and there ought to be a relation to their symplectic structures through understanding the moduli stack of objects in the Aug_+-category [72] as a wild character variety [6, 85]. It should be noted that our diagrammatic calculus, which we refer to as *algebraic weaves*, provides a combinatorial and explicit approach to these stratifications. In addition, the strata are compatible with the holomorphic symplectic structure, the open toric charts admitting (exponential) holomorphic Darboux coordinates.

Finally, the manuscript illustrates several new connections to braid varieties and algebraic weaves that remain to be explored. For instance, a relation to cluster algebras is partially detailed at the end of the article. We believe that the calculus of algebraic weaves, as described in the present manuscript, may prove useful in the study of cluster algebras, character varieties (including irregular singularities), subword complexes and Rouquier complexes, see Section 7.

1.2. **Main Results.** Let γ be a positive n-braid word $[\gamma] \in \mathrm{Br}_n^+$, $\gamma = \sigma_{i_1} \cdots \sigma_{i_\ell}$, and $\pi \in \mathrm{GL}(n, \mathbb{C})$ a permutation matrix. Associated to these data, we consider the *braid variety*

$$X_0(\gamma; \pi) := \{(z_1, \dots, z_\ell) \ : \ B_\gamma(z_1, \dots, z_\ell)\pi \text{ is upper-triangular}\} \subseteq \mathbb{C}^\ell,$$

where the matrix $B_\gamma(z_1, \dots, z_\ell) \in \mathrm{GL}(n, \mathbb{C}[z_1, \dots, z_\ell])$ is defined to be the matrix product

$$B_\gamma(z_1, \dots, z_\ell) := B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell),$$

and the matrices $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ are defined by:

$$(B_i(z))_{jk} := \begin{cases} 1 & j = k \text{ and } j \neq i, i+1 \\ 1 & (j,k) = (i, i+1) \text{ or } (i+1, i) \\ z & j = k = i+1 \\ 0 & \text{otherwise;} \end{cases}, \qquad \text{i.e.} \quad B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

These matrices $B_i(z)$ are referred to as *braid matrices*, and the only the non-trivial (2×2)-block is at ith and $(i+1)$st rows. Braid matrices have appeared in a range of areas, starting with L. Euler's continuants [33], G. Stokes' study of irregular singularities [87] (see P. Boalch's [8]), M. Broué and J. Michel's work on Deligne-Lusztig varieties [11], P. Deligne's braid invariants [25], and more recently in T. Kálmán's study of the Legendrian Contact DGA [54] (see also [20]) and A. Mellit's results on the curious Lefschetz property for character varieties [68], among others.

From our definition above, it is simple to see that $X_0(\gamma; \pi)$ is isomorphic to $X_0(\gamma'; \pi)$ if $[\gamma] = [\gamma'] \in \mathrm{Br}_n$, i.e. if two positive words γ, γ' represent the same n-braid, the resulting braid varieties are isomorphic, hence the name. In the course of the article, the permutation (matrix) π will often be the identity $\pi = \mathrm{Id} = e \in S_n$ or $\pi = w_0 = (n \ n-1 \ \dots \ 1) \in S_n$. Let $\Delta \in \mathrm{Br}_n^+$ be a positive braid lift of the permutation w_0, i.e. Δ will be a braid word for the half-twist.[1]

The first result of the article establishes geometric properties of braid varieties, including the existence of a torus action and their relation to the Floer-theoretically defined augmentation varieties [9, 23, 72]. It reads as follows:

Theorem 1.1. *Let γ be a positive n-braid word $[\gamma] \in \mathrm{Br}_n^+$. Then the following statements hold:*

(i) $X_0(\gamma\Delta; 1) \simeq X_0(\gamma; w_0) \times \mathbb{C}^{\binom{n}{2}}$, *and $X_0(\gamma; w_0)$ is non-empty if and only if the Demazure product of γ equals w_0. In this case, $X_0(\gamma; w_0)$ is an irreducible complete intersection of dimension $\ell(\gamma) - \binom{n}{2}$, and $X_0(\gamma\Delta; 1)$ is an irreducible complete intersection of dimension $\ell(\gamma)$.*

Suppose that there exists a positive n-braid word β such that $\gamma = \beta\Delta$. Then:

(ii) *The braid variety $X_0(\beta\Delta; w_0)$, and thus $X_0(\beta\Delta^2; 1)$, is smooth.*

(iii) *There exists a free torus T-action on $X_0(\beta\Delta; w_0)$ such that the quotient algebraic variety $X_0(\beta\Delta; w_0)/T$ is smooth and holomorphic symplectic.*

[1] See Example 2.2 for our specific choice of positive braid word for Δ.

(iv) *There exists an isomorphism between* $X_0(\beta\Delta; w_0)/T$ *and an augmentation variety* $\mathrm{Aug}(\beta)$ *associated to the Legendrian link* $\Lambda(\beta)$*. In particular,* $\mathrm{Aug}(\beta)$ *is a holomorphic symplectic (smooth) affine variety.*[2]

In addition, the open Bott-Samelson variety $\mathrm{OBS}(\beta)$ *associated to* β *is isomorphic to the quotient*

$$\mathrm{OBS}(\beta) \cong \left(\mathrm{GL}(n, \mathbb{C}) \times X_0(\beta; 1)\right)/\mathcal{B},$$

where $\mathcal{B} \subseteq \mathrm{GL}(n, \mathbb{C})$ *is the Borel subgroup of upper-triangular matrices.* □

In Theorem 1.1.(iii), the dimension of the torus T does depend on the number of components in the closure of β, see Section 2 for details. The different varieties and the torus action featured in Theorem 1.1 are presented in the course of the article, and the proof of this theorem is obtained by gathering some the results we develop, such as Theorem 2.30, Theorem 2.6, Theorem 3.5 and Corollary 5.22. See also Section 4.3 for the definition of Demazure product, and note that the Demazure product of $\beta\Delta$ equals w_0 for any β.

Remark 1.2. In the development of this article, we learned that the authors of [37] were also studying holomorphic symplectic structures on augmentation varieties, working on a different construction with plabic graphs. Their work is yet to appear but, once it does, it would be interesting to compare these symplectic structures, as the constructions seem to be significantly different and a potential link between weaves and plabic graphs could be fruitful. □

Theorem 1.1 discusses the absolute aspects of braid varieties. The study of such varieties also relies crucially on their relative geometry: morphisms between different such braids varieties and, more generally, correspondences, yield interesting (and useful) results. In order to study this relative setting, we develop the diagrammatic calculus of *algebraic weaves*, which we summarize as follows.

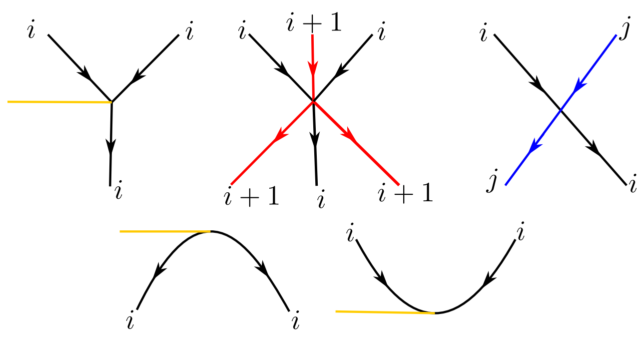
Let $\mathfrak{W}_n$ be the category defined as:

- **Objects**: $\mathrm{Ob}(\mathfrak{W}_n)$ are arbitrary positive braid words $\gamma = \sigma_{i_1} \cdots \sigma_{i_\ell}$, $[\gamma] \in Br_n^+$,
- **Morphisms**: $\mathrm{Hom}_{\mathfrak{W}_n}(\gamma, \gamma')$ are compositions of the four elementary moves

$$\sigma_i\sigma_i \to \sigma_i, \qquad \sigma_i\sigma_{i+1}\sigma_i \leftrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \sigma_i\sigma_j \to \sigma_j\sigma_i \ (|i-j|>1), \qquad \text{and} \qquad \sigma_i\sigma_i \leftrightarrow 1,$$

modulo certain relations, explicitly drawn in Section 4.2.

The morphisms in $\mathfrak{W}_n$ will be represented diagrammatically as certain planar graphs with edges decorated by simple transpositions s_i. (Namely, s_i are the Coxeter projections of the Artin braid generators σ_i, $1 \le i \le n$.) These planar graphs are referred to as *weaves*, following the notation in [19, Section 2], and $\mathfrak{W}_n$ will be called the *category of algebraic weaves.* The elementary moves above, i.e. the building blocks for morphisms, can be drawn as follows:



There is also a dual 6-valent vertex corresponding to $\sigma_i\sigma_{i-1}\sigma_i \to \sigma_{i-1}\sigma_i\sigma_{i-1}$ which we do not draw here. An algebraic weave, obtained by (vertically) concatenating the models above, represents a morphism from the braid word on the top to the braid word on the bottom. The composition of weaves

$$\mathrm{Hom}_{\mathfrak{W}_n}(\gamma, \gamma') \times \mathrm{Hom}_{\mathfrak{W}_n}(\gamma', \gamma'') \longrightarrow \mathrm{Hom}_{\mathfrak{W}_n}(\gamma, \gamma'')$$

is given by vertical stacking of these weave diagrams. See Figure 1 for an instance of a morphism.

[2]See Section 2.6 for the details on marked points.

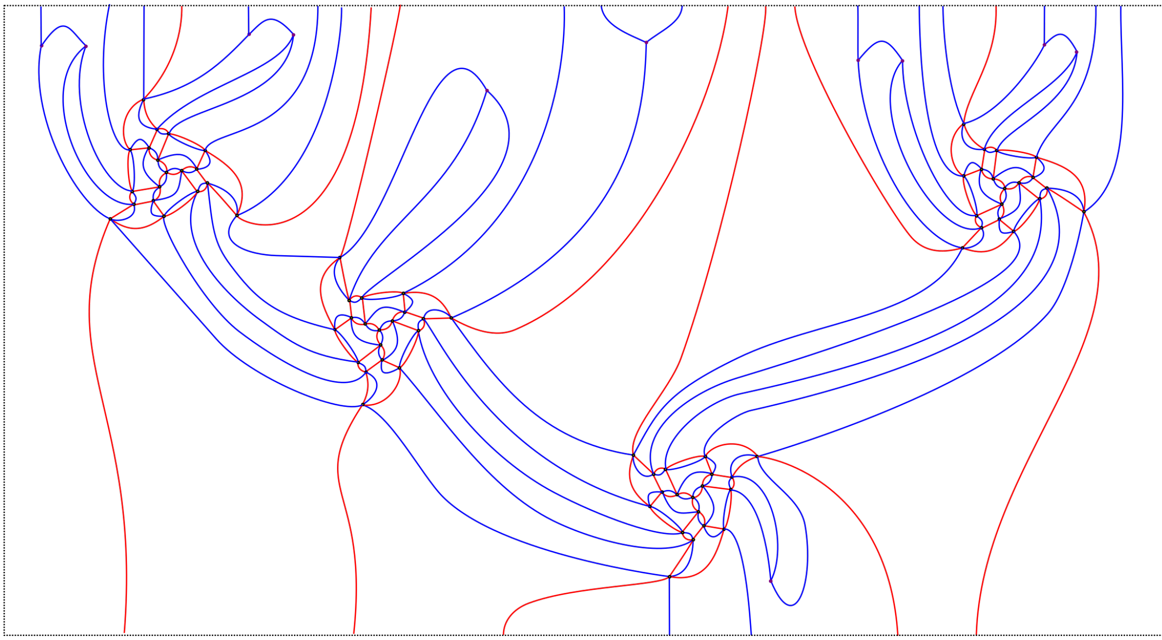
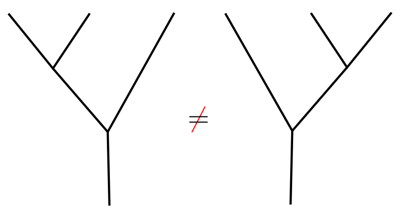


FIGURE 1. An algebraic weave in $\mathrm{Hom}_{\mathfrak{W}_n}(\gamma,\gamma')$ between the two positive 3-braids $\gamma=\sigma_1^3\sigma_2\sigma_1^3\sigma_1^2\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2\sigma_1^3$, on the top, and $\gamma'=\sigma_2^3\sigma_1^2\sigma_2^2$, on the bottom. The color code is that blue is labeled with the transposition s_1 and red is labeled with s_2.

Remark 1.3. Note that this diagrammatic category is in part similar to the categories appearing in Soergel calculus [30, 31], but differs in several key aspects. In particular, in the category of algebraic weaves there is no requirement that the two ways of getting from $\sigma_i\sigma_i\sigma_i$ to σ_i, via the moves $\sigma_i\sigma_i\to\sigma_i$, are equivalent:



The difference between these diagrams will be referred to as a *weave mutation*. □

Let $\mathfrak{C}$ be the category of algebraic varieties whose morphisms are correspondences [64]. The second result in this manuscript shows that the braid varieties, and their correspondences, provide a realization of the weave category $\mathfrak{W}_n$:

Theorem 1.4. *There exists a functor $\mathfrak{X}_0:\mathfrak{W}_n\longrightarrow\mathfrak{C}$ such that:*

(a) **Objects***: For a positive braid word $\gamma\in Ob(\mathfrak{W}_n)$, the functor $\mathfrak{X}_0$ associates the braid variety $\mathfrak{X}_0(\gamma):=X_0(\beta;w_0)$.*

(b) **Morphism***: For a weave $\Sigma\in\mathrm{Hom}_{\mathfrak{W}_n}(\beta_0,\beta_1)$, the functor associates a correspondence $\mathfrak{X}_0(\Sigma)$ between $X_0(\beta_0;w_0)$ and $X_0(\beta_1;w_0)$, such that correspondences $\mathfrak{X}_0(\Sigma)$ and $\mathfrak{X}_0(\Sigma')$ associated to equivalent weaves Σ,Σ' are isomorphic. (The algebraic variety $\mathfrak{X}_0(\Sigma)$ is in fact described as a certain moduli space governed by the weave Σ.)*

(c) **Composition***: Let $\Sigma_1\in\mathrm{Hom}_{\mathfrak{W}_n}(\beta_0,\beta_1)$, $\Sigma_2\in\mathrm{Hom}_{\mathfrak{W}_n}(\beta_1,\beta_2)$, and their composition $\Sigma\in\mathrm{Hom}_{\mathfrak{W}_n}(\beta_0,\beta_2)$, which is obtained by concatenation of Σ_1 and Σ_2. Then the composition of weaves corresponds to the diagram:*

$$\begin{tikzcd}[column sep=small]
 & & \mathfrak{X}_0(\Sigma) \arrow[dl] \arrow[dr] & & \\
 & \mathfrak{X}_0(\Sigma_1) \arrow[dl] \arrow[dr] & & \mathfrak{X}_0(\Sigma_2) \arrow[dl] \arrow[dr] & \\
X_0(\beta_0; w_0) & & X_0(\beta_1; w_0) & & X_0(\beta_2; w_0),
\end{tikzcd}$$

where the middle square is Cartesian.

(d) *Let $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta_0, \beta_1)$ be a weave with no caps, a cups and b trivalent vertices. Then the correspondence $\mathfrak{X}_0(\Sigma)$ defines an injective map*

$$\mathfrak{X}_0(\Sigma) : \mathbb{C}^a \times (\mathbb{C}^*)^b \times X_0(\beta_1; w_0) \hookrightarrow X_0(\beta_0; w_0).$$

Furthermore, the correspondences $\mathfrak{X}_0(\Sigma)$ are equivariant with respect to appropriate torus T-actions and, using Theorem 1.1.(iv), yield correspondences between augmentations varieties.

The proof of Theorem 1.4 occupies the majority of Section 5, the equivariance statement being discussed in Subsection 5.4.

Remark 1.5. The statements in Theorem 1.4.(a)-(c) are the algebraic analogues of the symplectic geometric results obtained by the first author in [19]. In short, [19] shows that any weave $\mathrm{Hom}_{\mathfrak{W}_n}(\beta_0, \beta_1)$ yields an exact Lagrangian cobordism $L(\Sigma) \subseteq (\mathbb{R}^4, \lambda_{\mathrm{st}})$ from the Legendrian link $\Lambda(\beta_1) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ to $\Lambda(\beta_0)$. Equivalent weaves give rise to Hamiltonian isotopic Lagrangians, and certain weaves yield embedded exact Lagrangian cobordisms. □

It should be noted that Theorem 1.4.(d) provides a unifying framework for many known stratifications, including the ruling stratification in augmentation varieties [36, 45, 46], and the stratification by walks in character varieties [68]. In particular, note that the algebraic variety $X_0(\Delta; w_0)$ is a point, and thus Theorem 1.4 implies the following:

Corollary 1.6. *Let $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\gamma, \Delta)$ be a weave with no caps, a cups and b trivalent vertices, $a, b \in \mathbb{N}$. Then the correspondence $\mathfrak{X}_0(\Sigma)$ yields an injective map*

$$\mathfrak{X}_0(\Sigma) : \mathbb{C}^a \times (\mathbb{C}^*)^b \hookrightarrow X_0(\gamma; w_0), \;\; 2a + b = \ell(\gamma) - \binom{n}{2}.$$

In fact, Theorem 5.21 will prove that weaves can be used to provide many stratifications of the braid varieties $X_0(\gamma; w_0)$, with strata of the form $\mathbb{C}^a \times (\mathbb{C}^*)^b$. In particular, weaves $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\gamma, \Delta)$ with no caps or cups yield (algebraic) toric charts $(\mathbb{C}^*)^{\ell(\gamma)-\binom{n}{2}} \subset X_0(\gamma; w_0)$ of maximal dimension, whose complement can also be stratified with weaves. These weaves, with no caps or cups, are referred to as *Demazure weaves.*

Remark 1.7. The manuscript also includes a new construction of weaves, coming from a class of labeled triangulations. This construction uses Demazure products in a crucial manner and provides a systematic (and combinatorial) mechanism to construct embedded exact Lagrangian fillings for Legendrian links $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ which are obtained as the closure of a positive braid β. □

Finally, complementing Theorem 1.1 and Theorem 1.4, we give a geometric interpretation to these toric charts associated to Demazure weaves $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta\Delta, \Delta)$, as follows.

First, we show in Section 2.3 that these charts can be combinatorially obtained by opening the crossings of the positive braid β. Indeed, Section 2.3 shows that there is an injective map

$$X_0(\beta'\Delta; w_0) \times \mathbb{C}^* \hookrightarrow X_0(\beta\Delta; w_0),$$

if the positive braid word β' is obtained from β by removing exactly one crossing. Therefore, opening the crossings in β one by one, in some order, yields a toric chart in $X_0(\beta\Delta; w_0)$. Different orders might yield identical or different toric charts. For instance, for a 2-strand braid $\beta = \sigma_1^n$, there are $n!$ possible orderings and one obtains a Catalan number C_n of toric charts. In particular, in this correspondence, each toric chart is obtained by exactly one 312-pattern avoiding permutation. Regarding this relation, between toric charts and openings of crossings, we establish the following result:

Theorem 1.8. *Let $[\beta] \in Br_n^+$ be a positive braid and $\beta = \sigma_{i_1} \cdot \sigma_{i_2} \cdot \ldots \cdot \sigma_{i_l}$ positive braid word. Consider an ordering $\rho \in S_l$ for the crossings of β. Then:*

(i) *There exists a (Demazure) weave Σ_ρ such that the sequence of crossing openings according to ρ is realized by the weave Σ_ρ. Conversely, any Demazure weave Σ is equivalent to opening crossings for some ordering $\rho \in S_l$, i.e. there exists $\rho \in S_l$ such that Σ is equivalent to Σ_ρ.*

(ii) *Two toric charts $C_1, C_2 \subseteq X_0(\beta\Delta; w_0)$ associated to different orderings of the crossings are represented by weaves Σ_1, Σ_2 such that Σ_1, Σ_2 are related by a sequence of mutations. In addition, the union of all such toric charts covers $X_0(\beta\Delta; w_0)$ up to codimension 2.*

The first item is proven in Lemma 5.6 and Theorem 5.8, and the proof of the codimension-2 cover is established in Theorem 2.18. The mutation equivalence of any weaves yielding toric charts follows from the more general Theorem 4.6, which states that, under technical conditions that are satisfied in the weaves pertaining to Theorem 1.8, any two Demazure weaves between the same two braid words are related by a sequence of equivalence moves and mutations. Note that Theorem 4.6 is a translation of a result of B. Elias [29] to our weaves framework.

Remark 1.9. Note that both the openings of crossings and mutations can be described in terms of braid words. Indeed, consider a braid word $\sigma_i u \sigma_j$ with $\sigma_i u = u\sigma_j$, i.e. (σ_i, σ_j) is deletion pair in the notation of [47]. Then, we can consider two different weaves:

$$\begin{tikzcd} & \sigma_i u \sigma_j \arrow[dl] \arrow[dr] & \\ \sigma_i \sigma_i u \arrow[d] & & u\sigma_j\sigma_j \arrow[d] \\ \sigma_i u \arrow[rr] & & u\sigma_j \end{tikzcd}$$

In this diagram, the left weave $\sigma_i u \sigma_i \to u\sigma_j$ corresponds to the opening of the crossing σ_i, and the right weave $\sigma_i u \sigma_i \to u\sigma_j$ to opening the crossing σ_j. Suppose that the positive braid word u can be broken down in the middle $u = u_1\sigma_k u_2$. Then we obtain:

$$\sigma_i u \sigma_j = \sigma_i u_1 \sigma_k u_2 \sigma_j, \ \sigma_i u_1 = u_1 \sigma_k, \ \sigma_k u_2 = u_2 \sigma_j,$$

and the two associated weaves are either related by a sequence of mutations, or they are equivalent. Theorem 1.8 ensures that any Demazure weave is equivalent to a weave composed from such pieces.□

We conclude by giving a topological interpretation to these toric charts. Given a weave Σ, we consider the associated *spectral curve* $L(\Sigma)$, constructed as a branched cover of the plane ramified along the trivalent vertices of the weave, following [19, 40, 85]. Then we show the following result:

Theorem 1.10. *Let β be a positive braid word, and $\Sigma \in Hom_{\mathfrak{W}_n}(\beta\Delta, \Delta)$ a Demazure weave. Then, the following two statements hold:*

(a) *The toric chart $C(\Sigma) \subseteq \mathrm{Aug}(\beta)$ corresponding to the weave Σ is naturally isomorphic to $\mathrm{Spec}(\mathbb{Z}[H_1(L(\Sigma), \mathbb{Z})]) \times S$, where S is a $2(k-1)$-dimensional symplectic torus that does not depend on Σ and k is the number of components of the closure of β.*

(b) *The restriction of the symplectic form to the toric chart $C(\Sigma) \cong \mathrm{Spec}(\mathbb{Z}[H_1(L(\Sigma), \mathbb{Z})])$ corresponds to the intersection form on $H_1(L(\Sigma), \mathbb{Z})$.* □

Remark 1.11. The article [37] endows $X_0(\beta\Delta, w_0)$ and $\mathrm{Aug}(\beta)$ with the structure of *cluster varieties*, with toric charts being cluster charts, and mutations corresponding to cluster mutations. To be precise, [37] only works in characteristic 2, but can be lifted to characteristic 0 by using the first author's upcoming work [20]. We expect that the cluster coordinates correspond to some natural bases in $H_1(L(\Sigma))$. (The difference between cluster X-coordinates and cluster A-coordinates is yet to be understood in the context of augmentation varieties.) It would be interesting to describe these bases and cluster coordinates directly from a weave, see Section 6 for a more detailed discussion. □

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2. Braid Varieties and Augmentation Varieties

In this section we introduce and start studying braid varieties. Part of Theorem 1.1 is proven in this section, with the holomorphic symplectic structure being discussed in Section 3. This section also discusses the torus actions on braid varieties and their quotients, which relate to augmentations varieties.

Notations for the braid group. Let $n \in \mathbb{N}$, the braid group Br_n on n-strands is presented with $n-1$ generators σ_i, $i \in [1, n-1]$, and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, i,j \in [1, n-1]. \tag{2.1}$$

In this article, we mainly work with the positive braid monoid $\mathrm{Br}_n^+ \subseteq \mathrm{Br}_n$ generated by the nonnegative powers of the generators σ_i, $i \in [1, n-1]$. By definition, a (positive) braid word is a product expression of non-negative powers of the generators σ_i where no relations are being applied. For instance, the two braid words $\sigma_1\sigma_2\sigma_3\sigma_1$ and $\sigma_2\sigma_1\sigma_2\sigma_3$ are distinct as braid words and represent the same element $[\sigma_1\sigma_2\sigma_3\sigma_1] = [\sigma_2\sigma_1\sigma_2\sigma_3] \in \mathrm{Br}_4^+$.

The symmetric group S_n is the Coxeter group associated to Br_n: it is generated by the transpositions $s_i = (i\ i+1)$, subject to relations (2.1) above and the additional relation $s_i^2 = 1$, for all $i \in [1, n-1]$. By definition, a reduced expression for a permutation $w \in S_n$ is a minimal length expression for the element w as a product of the generators s_i, $i \in [1, n-1]$; the length $\ell(w)$ is defined as the length of such reduced expression. It is well-known that any two reduced expressions are related by a sequence of braid moves (2.1). Therefore, one can define a positive braid lift of $w \in S_n$ to Br_n^+ by choosing an arbitrary reduced expression and replacing each generator s_i with the generator σ_i, for each $i \in [1, n-1]$. We will refer to such positive braid lifts as reduced braid words. To ease notation, we interchangeably use σ_i, s_i, and sometimes simply i for the braid group generators, $i \in [1, n-1]$.

2.1. Braid matrices and braid varieties.

Braid varieties are affine algebraic varieties cut out by matrix equations. Their definition relies on the following notion:

Definition 2.1. Let $n \in \mathbb{N}$, $i \in [1, n-1] \in \mathbb{N}$ and z a (complex) variable. The braid matrix $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ is defined as

$$(B_i(z))_{jk} := \begin{cases} 1 & j = k \text{ and } j \neq i, i+1 \\ 1 & (j,k) = (i, i+1) \text{ or } (i+1, i) \\ z & j = k = i+1 \\ 0 & \text{otherwise;} \end{cases}, \qquad \text{i.e.} \quad B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

Given a positive braid word $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathrm{Br}_n^+$ and $z_1, \ldots, z_r$ complex variables, we define the braid matrix $B_\beta(z_1, \ldots, z_r) \in \mathrm{GL}(n, \mathbb{C}[z_1, \ldots, z_r])$ to be the product

$$B_\beta(z_1, \ldots, z_r) = B_{i_1}(z_1) \cdots B_{i_r}(z_r).$$

□

For instance, it follows from Definition 2.1 that $B_\beta(0, \ldots, 0)$ is simply the permutation matrix associated to the Coxeter projection $\pi(\beta) \in S_n$. Thus, in a sense, braid matrices are deformations of permutation matrices. It is a simple computation to verify the two relations:

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1), \quad \forall i \in [1, n-2], \tag{2.2}$$

and $B_i(z_1)B_j(z_2) = B_j(z_2)B_i(z_1)$, for $|i-j| \geq 2$. Here is a first pair of useful examples:

Example 2.2. *Consider the positive braid word $\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1$, which represents a half-twist. Its associated braid matrix is*

$$B_\Delta\left(z_1, \ldots, z_{\binom{n}{2}}\right) = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & \ldots & 1 & z_1 \\ \vdots & \ldots & \ldots & z_{n-2} \\ 1 & z_{\binom{n}{2}} & \cdots & z_{n-1} \end{pmatrix}.$$

Let $\Delta' \in \mathrm{Br}_n^+$ be any *positive braid lift of the longest element w_0 of S_n. It then follows from the braid relation* (2.2) *that*

$$B_{\Delta'}\left(z_1, \ldots, z_{\binom{n}{2}}\right) = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & \ldots & 1 & z_{2,n} \\ \vdots & \ldots & \ldots & z_{n-1,n} \\ 1 & z_{n,2} & \cdots & z_{nn} \end{pmatrix},$$

where the $z_{i,j}$ are algebraically independent polynomials in $\mathbb{C}[z_1, \ldots, z_{\binom{n}{2}}]$. □

Lemma 2.3. *Let $\Delta^2 \in \mathrm{Br}_n^+$ represent the full-twist braid, i.e. the square of the positive braid lift of $w_0 \in S_n$ to the braid group. Then its braid matrix can be decomposed as*

$$B_{\Delta^2}\left(z_1, \ldots, z_{\binom{n}{2}}, w_1, \ldots w_{\binom{n}{2}}\right) = LU = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ c_{21} & 1 & \ldots & 0 \\ \vdots & \cdots & \ddots & 0 \\ c_{n1} & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \ldots & u_{1n} \\ 0 & 1 & \ldots & u_{2n} \\ 0 & \cdots & \ddots & u_{n-1,n} \\ 0 & \cdots & \cdots & 1 \end{pmatrix},$$

where $c_{ij} \in \mathbb{C}[z_1, \ldots, z_{\binom{n}{2}}]$ and $u_{ij} \in \mathbb{C}[w_1, \ldots, w_{\binom{n}{2}}]$ are algebraically independent.

Proof. By Example 2.2, $B_\Delta = Lw_0 = w_0U$. Hence $B_{\Delta^2} = B_\Delta B_\Delta = Lw_0w_0U = LU$. □

Let us now use braid matrices to define the central object of interest in this manuscript:

Definition 2.4. Let $\beta = \sigma_{i_1}\cdots\sigma_{i_r} \in \mathrm{Br}_n^+$ be a positive braid word. The braid variety $X_0(\beta) \subseteq \mathbb{C}^r$ associated to β is the affine closed subvariety given by

$$X_0(\beta) := \{(z_1, \ldots, z_r) : B_\beta(z_1, \ldots, z_r) \text{ is upper-triangular}\} \subseteq \mathbb{C}^r.$$

Let $\pi \in S_n$ be considered as a permutation matrix. We define the braid variety $X_0(\beta; \pi) \subseteq \mathbb{C}^r$ as

$$X_0(\beta;\pi) := \{(z_1, \ldots, z_r) : B_\beta(z_1, \ldots, z_r)\pi \text{ is upper-triangular}\} \subseteq \mathbb{C}^r.$$

It follows from the braid relation (2.2) that different presentations of the same braid $[\beta] \in \mathrm{Br}_n$ yield algebraically isomorphic braid varieties. □

Let us give some simple examples of braid varieties.

Example 2.5. *Consider the positive braid associated to the full twist $\beta = \Delta^2$. Lemma 2.3 implies that $X_0(\Delta^2)$ is given by the equations $c_{ij} = 0$, and thus the braid variety is the affine space $X_0(\Delta^2) \cong \mathbb{C}^{\binom{n}{2}}$. Similarly, Example 2.2 implies that the braid variety $X_0(\Delta; w_0) = \{\mathrm{pt}\}$ is a point.* □

In fact, the computation in Example 2.2 shows that $X_0(\beta; w_0)$ admits a closed embedding into $X_0(\beta \cdot \Delta)$, and the image is given by the equations $z_{ij} = 0$. In general, if $\Pi \in \mathrm{Br}_n^+$ is a positive lift of a permutation $\pi \in S_n$ then $X_0(\beta;\pi)$ embeds into $X_0(\beta \cdot \Pi)$. Let us now establish the general dimension and smoothness for braid varieties:

Theorem 2.6. *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid of length $\ell(\beta)$. Then, the braid varieties $X_0(\beta \cdot \Delta; w_0)$ and $X_0(\beta \cdot \Delta^2)$ are smooth of dimension $\ell(\beta)$ and $\ell(\beta) + \binom{n}{2}$, respectively. In addition, $X_0(\beta \cdot \Delta^2) \simeq X_0(\beta \cdot \Delta; w_0) \times \mathbb{C}^{\binom{n}{2}}$.*

Proof. The variety $X_0(\beta\cdot\Delta^2)$ is defined by the condition that $B_{\beta\cdot\Delta^2}$ is an upper triangular matrix. By Lemma 2.3, we get

$$B_{\beta\cdot\Delta^2} = B_\beta B_{\Delta^2} = B_\beta LU = B_{\beta\cdot\Delta} w_0 U.$$

This is upper-triangular if and only if $B_{\beta\cdot\Delta}w_0$ is upper-triangular, which is precisely the condition defining $X_0(\beta\cdot\Delta; w_0)$. Therefore, $X_0(\beta\cdot\Delta^2)\simeq X_0(\beta\cdot\Delta;w_0)\times\mathbb{C}^{\binom{n}{2}}$, with $\mathbb{C}^{\binom{n}{2}}$ being the coordinates on the upper uni-triangular matrix U. Next, note that $B_{\beta\cdot\Delta}w_0 = B_\beta L$ and we can write $B_\beta L = U'$, then

$$B_\beta^{-1} = L(U')^{-1}.$$

Note that the existence of an LU decomposition $M = LU''$ is an open condition on M, namely the non-vanishing of principal minors; also, if an LU decomposition exists, it is unique provided that L has 1s on the diagonal. Therefore $X_0(\beta\cdot\Delta;w_0)$ is isomorphic to an open subset in the affine space $\mathbb{C}^{\ell(\beta)}$. Hence, it is smooth of dimension $\dim X_0(\beta\cdot\Delta;w_0)=\ell(\beta)$, and $X_0(\beta\cdot\Delta^2)$ is also smooth of dimension $\dim X_0(\beta\cdot\Delta^2)=\ell(\beta)+\binom{n}{2}$, as required. □

Note that a similar smoothness result was proved in [84, Theorem 2.30]. The braid varieties associated to 2-stranded braids $\beta\in\mathrm{Br}_2^+$ are smooth varieties whose equations closely relate to Euler's continuants [33]. Let us consider the following instance:

Example 2.7. (*Trefoil Knot*) *Consider $\beta=\sigma_1^3\in Br_2^+$, whose (-1)-framed closure yields the (max-tb) right-handed trefoil knot. The braid variety $X_0(\sigma_1^5)=X_0(\sigma_1^3\cdot\Delta^2)$ is defined by the equation:*

$$B(z_1)B(z_2)B(z_3)B(z_4)B(z_5) \text{ is upper-triangular.}$$

This condition can written as

$$B(z_1)B(z_2)B(z_3)\begin{pmatrix}1&0\\z_4&1\end{pmatrix}\begin{pmatrix}1&z_5\\0&1\end{pmatrix} \text{ is upper-triangular,}$$

and equivalently

$$B(z_1)B(z_2)B(z_3)\begin{pmatrix}1&0\\z_4&1\end{pmatrix}=\begin{pmatrix}z_2+(z_2z_3+1)z_4 & z_2z_3+1\\ z_1z_2+(z_1+(z_1z_2+1)z_3)z_4+1 & z_1+(z_1z_2+1)z_3\end{pmatrix} \text{ upper-triangular.}$$

Note that we have

$$\begin{pmatrix}1&0\\z_4&1\end{pmatrix}=\begin{pmatrix}0&1\\1&z_4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&z_4\end{pmatrix}w_0,$$

and thus the condition above, in the coordinates $(z_1,z_2,z_3,z_4)\in\mathbb{C}^4$, is in fact the equation for $X_0(\sigma_1^3\cdot\Delta;w_0)$. This readily implies that $X_0(\sigma_1^3\cdot\Delta^2)$ is isomorphic to $X_0(\sigma_1^3\cdot\Delta;w_0)$ times an affine line $\mathbb{C}=\mathrm{Spec}(\mathbb{C}[z_5])$, as generally proven in Theorem 2.6. It thus suffices to understand $X_0(\sigma_1^3\cdot\Delta;w_0)$. For that, consider the equation above:

$$X_0(\sigma_1^3\cdot\Delta;w_0)=\{(z_1,z_2,z_3,z_4)\in\mathbb{C}^4 : 1+z_1z_2+z_4(z_1+z_3+z_1z_2z_3)=0\}\subseteq\mathbb{C}^4, \tag{2.3}$$

which cuts out a hypersurface, and should be smooth according to Theorem 2.6. Indeed, note that we must have $z_1+z_3+z_1z_2z_3\neq 0$, otherwise the defining Equation 2.3 would imply $1+z_1z_2=0$, and in these constraints $z_1=z_1+z_3(z_1z_2+1)=z_1+z_3+z_1z_2z_3=0$. This is a contradiction; thus, $z_1+z_3+z_1z_2z_3\neq 0$ in $X_0(\sigma_1^3\cdot\Delta;w_0)$. In consequence, $X_0(\sigma_1^3\cdot\Delta;w_0)$ is isomorphic to the open subset

$$X_0(\sigma_1^3\cdot\Delta;w_0)=\{(z_1,z_2,z_3)\in\mathbb{C}^3 : (z_1+z_3+z_1z_2z_3)\neq 0\}\subseteq\mathbb{C}^3,$$

since the coordinate z_4 can be obtained uniquely from any points $(z_1,z_2,z_3)\in\mathbb{C}^3$ in this subset. This shows that $X_0(\sigma_1^3\cdot\Delta;w_0)$ is smooth.

In fact, this provides a rather simple description for this braid variety: it is the open set foliated by the smooth hypersurfaces $(z_1+z_3+z_1z_2z_3)=a$, $a\in\mathbb{C}^*$, which are (in)famous for a variety of reasons, including their appearance in the isomonodromic problem for Painlevé I with a 5/2-singularity at infinity, the study of the augmentation variety for the trefoil, and mirror symmetry for the 2-torus with two transversal Lagrangian 2-disks attached [15, 18]*, being an instance of a self-mirror affine surface. In particular, an arboreal Lagrangian 3-skeleton for the Stein 3-manifold $X_0(\sigma_1^3\cdot\Delta;w_0)$ is depicted in Figure 2.* □

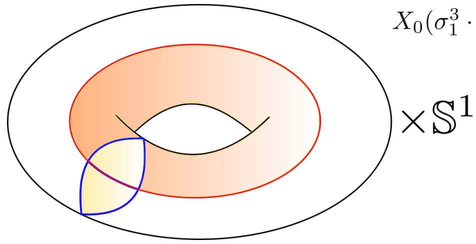


FIGURE 2. An arboreal Lagrangian 3-skeleton for the Stein structure of the (complex) 3-dimensional braid variety $X_0(\sigma_1^3\cdot\Delta;w_0)$ associated to the trefoil braid $\beta=\sigma_1^3\in\mathrm{Br}_2^+$. The skeleton consists of a circle $\mathbb{S}^1$ times a 2-torus $\mathbb{T}^2$ with two 2-disks attached to it along the curves $(1,0),(0,1)\in H_1(\mathbb{T}^2,\mathbb{Z})$.

Remark 2.8. In the case of positive braids associated to algebraic knots $K\subseteq\mathbb{R}^3$, the braid varieties can be similarly described symplectically using the arboreal skeleta constructed in [15]. In general, following the lines of Example 2.7, the braid varieties for $(2,n)$-torus links can be understood rather immediately, their computation often reducing to the understanding of a certain hypersurface. □

Note that the braid variety $X_0(\sigma_1^3\cdot\Delta;w_0)$ in Example 2.7 contains an $\mathbb{S}^1$-factor in its Lagrangian skeleton, and the Stein surface $X=\{z_1+z_3+z_1z_2z_3=1\}\subseteq\mathbb{C}^3$ is described by the Lagrangian 2-skeleton given by Figure 2 without the $\mathbb{S}^1$-factor. Since we can write

$$X_0(\sigma_1^3\cdot\Delta;w_0)\cong\{(z_1,z_2,z_3,t):(z_1+z_3+z_1z_2z_3)t=1\}\subseteq\mathbb{C}^3\times\mathbb{C}^*,$$

there exists a $\mathbb{C}^*$-action on $X_0(\sigma_1^3\cdot\Delta;w_0)$ whose quotient yields the affine surface X. The feature of admitting certain (complex) torus actions with interesting quotients is a general property of braid varieties, as we will now see.

2.2. **Torus actions on braid varieties.** Let $[\beta]\in\mathrm{Br}_n^+$ be a positive braid with a fixed positive braid word β. Consider the (Cartan) subgroup $\mathbb{T}\cong(\mathbb{C}^*)^n\subseteq\mathrm{GL}(n,\mathbb{C})$ of diagonal matrices. In this section we construct an algebraic T-action on the braid variety $X_0(\beta)$. First, we observe that

$$\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}\begin{pmatrix}0&1\\1&z\end{pmatrix}=\begin{pmatrix}0&1\\1&\frac{t_2}{t_1}z\end{pmatrix}\begin{pmatrix}t_2&0\\0&t_1\end{pmatrix}. \tag{2.4}$$

Let $D_{\mathbf{t}}=\mathrm{diag}(t_1,\dots,t_n)\in\mathbb{T}$ be a diagonal matrix. In general, we have $D_{\mathbf{t}}B_i(z)=B_i\left(\frac{t_{i+1}}{t_i}z\right)D_{s_i(\mathbf{t})}$, for s_i the Coxeter projection of σ_i. Thus

$$D_{\mathbf{t}}B_{i_1}(z_1)\cdots B_{i_r}(z_r)=B_{i_1}(c_1z_1)\cdots B_{i_r}(c_rz_r)D_{w(\mathbf{t})}, \tag{2.5}$$

where $r=\ell(\beta)$, $c_k=t_{w_k(i_k+1)}t^{-1}_{w_k(i_k)}$, $w_k=s_{i_1}\cdots s_{i_{k-1}}$ and $w=w_{r+1}$ is the permutation corresponding to β. The torus actions we study are defined as follows:

Definition 2.9. Let β be a positive n-braid word of length $r=\ell(\beta)$. The torus $\mathbb{T}$-action of $\mathbb{T}\cong(\mathbb{C}^*)^n$ on affine space $\mathbb{C}^{\ell(\beta)}$ is given by

$$\mathbf{t}.(z_1,\dots,z_r):=(c_1z_1,\dots,c_rz_r),\quad \mathbf{t}\in\mathbb{T},\quad (z_1,\dots,z_r)\in\mathbb{C}^r,$$

where c_i are defined as above, $i\in[1,r]$. Note that this torus $\mathbb{T}$-action preserves the braid variety $X_0(\beta)\subseteq\mathbb{C}^r$ thanks to relation 2.5.

By definition, the T-torus action $T\times X_0(\beta)\longrightarrow X_0(\beta)$ on the braid variety $X_0(\beta)$ is the quotient of the restriction of the above $\mathbb{T}$-action to $X_0(\beta)$ by the diagonal subtorus $\mathbb{C}^*\subseteq\mathbb{T}$, i.e. $T:=\mathbb{T}/\mathbb{C}^*_{diag}\cong(\mathbb{C}^*)^n/\mathbb{C}^*\cong(\mathbb{C}^*)^{n-1}$. The $\mathbb{T}$-action descends to the T-action quotient since the diagonal subtorus $(t,\dots,t)\subseteq\mathbb{T}$ acts trivially on $X_0(\beta)$. □

The torus action on $\mathbb{C}^r$ in Definition 2.9 depends on the choice of braid word β. Nevertheless, if β and β' are two positive presentations of the same braid, $[\beta]=[\beta']\in\mathrm{Br}_n^+$, then there exists an algebraic isomorphism $X_0(\beta)\cong X_0(\beta')$ which is equivariant with respect to this torus action. In addition, the T-action preserves the product decomposition

$$X_0(\beta\cdot\Delta^2)\cong X_0(\beta\cdot\Delta;w_0)\times\mathbb{C}^{\binom{n}{2}}$$

established in Theorem 2.6. Let us study this T-action on braid varieties $X_0(\beta)$.

Let $c(\beta)$ be the number of cycles in the cycle decomposition of the Coxeter projection $\pi(\beta) \in S_n$, i.e. the number of cycles of β understood as a permutation. The braid $[\beta] \in \mathrm{Br}_n$ closes up[3] to a knot in $\mathbb{R}^3$ if and only if $c(\beta) = 1$, and there are $(n-1)!$ such permutations $\pi(\beta) \in S_n$. For a braid associated to a knot, we have the following result:

Lemma 2.10. *Let β be a positive braid word, $[\beta] \in \mathrm{Br}_n^+$, with $c(\beta) = 1$. Then the torus T-action of $T \cong (\mathbb{C}^*)^{n-1}$ on the braid variety $X_0(\beta)$ is free.*

Proof. Let $(z_1, \ldots, z_r) \in X_0(\beta)$ and assume $\mathbf{t}.(z_1, \ldots, z_r) = (z_1, \ldots, z_r)$ for some $\mathbf{t} \in (\mathbb{C}^*)^n$. In particular, we have that $B_\beta(z) = B_\beta(\mathbf{t}.z)$. Thanks to Equality (2.5), we have that $D_{\mathbf{t}} B_\beta(z) D^{-1}_{w(\mathbf{t})} = B_\beta(z)$. Since $z \in X_0(\beta)$, the matrix $B_\beta(z)$ is upper triangular, and therefore its diagonal entries must be nonzero, as $\det(B_\beta(z)) = \pm 1$. From the equation

$$D_{\mathbf{t}} B_\beta(z) D^{-1}_{w(\mathbf{t})} = B_\beta(z),$$

it follows that $t_i t^{-1}_{w(i)} = 1$ for every $i = 1, \ldots, n$. Given that $c(\beta) = 1$, we must have that $t_i = t_j$ for all i, j and the result follows. $\square$

Corollary 2.11. *Let β be a positive braid word, $[\beta] \in \mathrm{Br}_n^+$, with $c(\beta) = 1$. Then the quotients of the braid varieties $X_0(\beta \cdot \Delta^2)/T$ and $X_0(\beta \cdot \Delta; w_0)/T$ are smooth and of dimension $\ell(\beta) + \binom{n}{2} - n + 1$ and $\ell(\beta) - n + 1$, respectively.* $\square$

The hypothesis $c(\beta) = 1$ in Lemma 2.10 is needed, as the T-actions on the braid varieties will in general fail to be free. For instance, consider the 2-stranded braid $\beta = \sigma_1^4$, associated to the Hopf link under the (-1)-framed closure, and its braid variety

$$X_0(\beta) \cong \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1 + z_3(1 + z_1 z_2) = 0\}.$$

The torus T-action scales z_1 and z_3 by $t \in T \cong \mathbb{C}^*$, and scales z_2 and z_4 by t^{-1}. Hence, it has a fixed point $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0) \in X_0(\beta)$. The following remark explains how to proceed in the case that $c(\beta) \neq 1$.

Remark 2.12. Consider a positive braid word β such that $[\beta] \in \mathrm{Br}_n^+$ closes up to a link with k connected components, i.e. $c(\beta) = k$. Then, we can construct several subtori of T that act freely on $X_0(\beta)$ as follows. Let $O_1, \ldots, O_k$ be the orbits for the action of w on $\{1, \ldots, n\}$, where w is the permutation associated to β. Choose a representative $j_m \in O_m$ for every $m = 1, \ldots, k$. Let $T' \subseteq T$ be the torus defined by the extra equations $t_{j_1} = t_{j_2} = \cdots = t_{j_k}$, noting that $T' \cong (\mathbb{C}^*)^{n-k}$. The same argument as in the proof of Lemma 2.10 shows that T' acts freely on $X_0(\beta)$. Note that we obtain that the quotient braid variety $X_0(\beta \cdot \Delta; w_0)/T'$ is a smooth variety of dimension $\ell(\beta) - n + k$, and a similar result holds for the quotient $X_0(\beta \cdot \Delta^2)/T'$. $\square$

This concludes the discussion on the torus T-action on $X_0(\beta)$. The geometric structures discussed during the article, e.g. stratifications and holomorphic symplectic structures, are compatible with these torus T-actions, and will be studied for the braid varieties $X_0(\beta)$ and their quotients $X_0(\beta)/T$.

2.3. **Toric charts in braid varieties.** In this subsection, we construct a codimension-0 toric chart $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta; w_0)$ associated to an (arbitrary) ordering $\tau(\beta) \in S_{l(\beta)}$ of the crossings of the positive braid word β. For that, consider two n-braid words

$$\beta = \sigma_{i_1}\sigma_{i_2} \cdot \ldots \cdot \sigma_{i_{k-1}}\sigma_{i_k}\sigma_{i_{k+1}} \cdot \ldots \cdot \sigma_{i_l}, \quad \beta' = \sigma_{i_1}\sigma_{i_2} \cdot \ldots \cdot \sigma_{i_{k-1}}\sigma_{i_{k+1}} \cdot \ldots \cdot \sigma_{i_l},$$

i.e. β' is obtained from β by removing the kth crossing σ_{i_k}. We will construct a rational map $X_0(\beta \cdot \Delta^2) \dashrightarrow X_0(\beta' \cdot \Delta^2) \times \mathbb{C}^*$ that identifies the latter variety with an explicit open set in $X_0(\beta \cdot \Delta^2)$.

First, we start with the following simple:

Lemma 2.13. *Let L and U be two invertible lower- and upper-triangular matrices, respectively. Then there exist lower- and upper-triangular matrices $\widetilde{L}$ and $\widetilde{U}$ such that*

$$B_i(z)U = \widetilde{U} B_i\left(\frac{u_{i+1,i+1}z + u_{i,i+1}}{u_{i,i}}\right), \ LB_i(z) = B_i\left(\frac{u_{i+1,i+1}z + l_{i+1,i}}{l_{i,i}}\right)\widetilde{L}.$$

[3]Either through the rainbow or (-1)-framed closure.

Moreover, $\widetilde{u}_{i,i+1} = \widetilde{l}_{i+1,i} = 0$ *and* $\widetilde{u}_{k,k} = u_{s_i(k),s_i(k)}$ *for every* k.

Proof. We prove the statement for the upper-triangulation matrix U, the case of L is proven analogously. By setting $\widetilde{u}_{i,i+1} = 0$, one obtains the system of equations:

$$\widetilde{u}_{j,i} = u_{j,i+1},\ \widetilde{u}_{j,i} + \widetilde{u}_{j,i+1}(z + u_{i,i+1}) = u_{j,i+1},\ (j < i),$$
$$\widetilde{u}_{i,j} = u_{i+1,j},\ \widetilde{u}_{i+1,j} = u_{i,j} + z u_{i+1,j},\ (j > i+1).$$

This system can be readily solved. Indeed, we first deduce that $\widetilde{u}_{k,k} = u_{k,k}$ if $k \neq i, i+1$. From the equations $u_{i,i} + z u_{i+1,i} = \widetilde{u}_{i+1,i+1}$, we obtain $\widetilde{u}_{i+1,i+1} = u_{i,i}$. Finally, one concludes that $\widetilde{u}_{i,i} = u_{i+1,i+1}$. □

They key algebraic equality that incarnates opening a crossing σ_i in a positive braid word, in terms of braid matrices, reads

$$B_i(z) = U_i(z) D_i(z) L_i(z), \tag{2.6}$$

where the variable $z \in \mathbb{C}^*$ associated to that crossing σ_i is now assumed to be non-zero. In this equation, we have used the matrices

$$U_i(z) := \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}, \quad D_i(z) := \begin{pmatrix} -z^{-1} & 0 \\ 0 & z \end{pmatrix}, \quad L_i(z) := \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix},$$

understood as being the (2×2)-block matrices placed in i-th and $(i+1)$-st row and column. Let us now illustrate how the process of opening a crossing occurs at the level of general braid matrices, as follows. Consider the positive braid word $\beta = \beta_1 \sigma_i \beta_2$ and the braid word $\beta' = \beta_1 \beta_2$ obtained by opening (i.e. removing) the explicit crossing σ_i between β_1, β_2. In order to apply Equation 2.6 we must assume that the variable z associated to the crossing σ_i is non-vanishing, and we always do so. Then we write

$$B_\beta = B_{\beta_1}(z_1, \ldots, z_{r-1}) B_i(z) B_{\beta_2}(z_{r+1}, \ldots, z_\ell) = B_{\beta_1} U_i(z) D_i(z) L_i(z) B_{\beta_2},$$

and use both Equation (2.4) and Lemma 2.13 to slide the middle matrices to the sides, U, D to the left and L to the right. This results in a decomposition of the form

$$B_\beta = U' D' B_{\beta_1}(z'_1, \ldots, z'_{r-1}) B_{\beta_2}(z'_{r+1}, \ldots, z'_\ell) L' = U' D' B_{\beta'}(z'_1, \ldots, z'_{r-1}, z'_{r+1}, \ldots, z'_\ell) L'$$

where U', L' and D' are some explicit upper (lower) uni-triangular and diagonal matrices, respectively, and $z'_1, \ldots, z'_{r-1}, z'_{r+1}, \ldots, z'_\ell$ are polynomial functions on $z_1, \ldots, z_{r-1}, z_r^{\pm 1}, z_{r+1}, \ldots, z_\ell$. Note that $B_\beta(z) L_1$ is upper-triangular for some lower-triangular matrix L_1 if and only if $B_{\beta'}(z') L' L_1$ is upper-triangular. These are the first ingredients for the construction of the rational map $\Omega_{\sigma_i} : X_0(\beta\Delta^2) \dashrightarrow X_0(\beta'\Delta^2) \times \mathbb{C}^*$.

For the second ingredient, we consider a point $(z_1, \ldots, z_\ell, c_{ij}) \in X_0(\beta \cdot \Delta^2)$. By Theorem 2.6, this is equivalent to $B_\beta(z) L(c_{ij})$ being upper triangular. Now we open a crossing, so we assume $z_i \neq 0$ is non-vanishing: using the decomposition above we obtained that $B_{\beta'}(z'_1, \ldots, z'_{r-1}) L' L(c_{ij})$ is upper triangular. Since $L' L(c_{ij})$ is lower triangular with 1's in the diagonal, we can write $L' L(c_{ij}) = L(c'_{ij})$, where c'_{ij} are polynomial functions on $z_r^{-1}, z_{r+1}, \ldots, z_\ell, c_{ij}$. These polynomial functions are the second ingredient. In summary, we obtain:

Definition 2.14. Consider the positive braid word $\beta = \beta_1 \sigma_i \beta_2$ of length $\ell = l(\beta)$, $\beta' = \beta_1 \beta_2$, and suppose that the complex variable z_i associated to the (middle) crossing σ_i is non-vanishing. By definition, the rational map Ω_{σ_i} associated to opening the crossing σ_i is

$$\Omega_{\sigma_i} : X_0(\beta\Delta^2) \dashrightarrow X_0(\beta'\Delta^2) \times \mathbb{C}^*, (z_1, \ldots, z_\ell, c_{ij}) \mapsto (z'_1, \ldots, z'_{r-1}, z'_{r+1}, \ldots, z'_\ell, c'_{ij}, z_r^{-1}),$$

where $z'_i \in \mathbb{C}[z_1, \ldots, z_{r-1}, z_r^{\pm}, z_{r+1}, \ldots, z_\ell], c'_{ij} \in \mathbb{C}[z_r^{-1}, z_{r+1}, \ldots, z_\ell, c_{ij}]$ are the polynomial functions defined as above. □

In the same notation and hypothesis as above, we have that:

Lemma 2.15. *The rational map*

$$\Omega_{\sigma_i} : X_0(\beta\Delta^2) \dashrightarrow X_0(\beta'\Delta^2) \times \mathbb{C}^*$$

restricts to an isomorphism between the open locus $\{z_r \neq 0\} \subseteq X_0(\beta \cdot \Delta^2)$ *and* $X_0(\beta' \cdot \Delta^2) \times \mathbb{C}^*$.

Proof. From the construction, see e.g. Lemma 2.13, if we know $z'_1, \dots, z'_{r-1}, z'_{r+1}, \dots, z'_\ell$ and z_r then we can reconstruct $z_1, \dots, z_\ell$, provided $z_r \neq 0$. It remains to show that, if we also know c'_{ij} then we can reconstruct c_{ij} as well. For that, we just notice that we can reconstruct L', and we have the equation $L(c_{ij}) = (L')^{-1} L(c'_{ij})$. □

There are two fundamental properties of these rational maps Ω_{σ_i}: they can be iterated, and they are compatible with the torus T-action. This is summarized in the following

Proposition 2.16. *Let β be a positive n-braid word. For each ordering $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings of β, there exists an open set $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta^2)$ such that:*

(i) $T_{\tau(\beta)} \cong (\mathbb{C}^*)^{\ell(\beta)} \times X_0(\Delta^2) = (\mathbb{C}^*)^{\ell(\beta)} \times \mathbb{C}^{\binom{n}{2}}$.

(ii) *$T_{\tau(\beta)}$ is given by the nonvanishing of Laurent polynomials in $z_{r_1}, z'_{r_2}, z''_{r_3}, \dots, z^{(\ell-1)}_{r_\ell}$; these latter variables can be taken as coordinates of the $(\mathbb{C}^*)^{\ell(\beta)}$-factor.*

(iii) *$T_{\tau(\beta)}$ is stable under the torus $(\mathbb{C}^*)^{n-1}$-action on $X_0(\beta \cdot \Delta^2)$.*

Proof. Given the discussion prior to the statement, the only assertion that needs a proof is the stability under the torus action. For that, we need to show that $z_{r_1}, z'_{r_2}, \dots, z^{(\ell-1)}_{r_\ell}$ are all homogeneous under the $(\mathbb{C}^\times)^{n-1}$-action. This will be proven in Lemma 2.22 and Lemma 2.23 below[4], and their corresponding analogues in the case of lower-triangular matrices. □

Proposition 2.16 and the relation between the braid varieties $X_0(\beta \cdot \Delta^2)$ and $X_0(\beta \cdot \Delta; w_0)$, as established in Theorem 2.6, readily imply the following result:

Corollary 2.17. *Let β be a positive n-braid word. For each ordering $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings of β, there exists an open set $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta; w_0)$ which is isomorphic to a torus $T_{\tau(\beta)} \cong (\mathbb{C}^*)^{\ell(\beta)}$ and stable under the torus $(\mathbb{C}^*)^{n-1}$-action $X_0(\beta \cdot \Delta; w_0)$.* □

The union of the toric charts $T_{\tau(\beta)}$ in Corollary 2.17, as $\tau(\beta) \in S_{\ell(\beta)}$ ranges through all the possible orderings, does not necessarily cover the entire variety $X_0(\beta \cdot \Delta; w_0)$. Fortunately, we can show that it does cover up to codimension-2:

Theorem 2.18. *Let β be a positive braid word. The complement*

$$X_0(\beta \cdot \Delta; w_0) \setminus \left(\bigcup_{\tau(\beta) \in S_{\ell(\beta)}} T_{\tau(\beta)} \right) \subseteq X_0(\beta \cdot \Delta; w_0)$$

has codimension at least 2.

Proof. Let us prove this by induction on the length $\ell(\beta) \in \mathbb{N}$. The base case, $\ell(\beta) = 0$ holds, as $X_0(\Delta; w_0) = \{\mathrm{pt}\}$, see Example 2.5. Note that for the case $\ell(\beta) = 1$, $\beta = \sigma_i$ for some $i \in [1, n-1]$, and $X_0(\beta\Delta; w_0)$ is defined by the condition that $B_i(z)^{-1}$ admits an LU-decomposition. (See the proof of Theorem 2.6.) This is equivalent to the non-vanishing $z \neq 0$ and thus $X_0(\beta \cdot \Delta; w_0) = \mathbb{C}^*$; thus the statement also holds.

For the induction step, we assume the statement to be true for length $l \in \mathbb{N}$ and suppose that $\ell(\beta) = \ell + 1$. Let $U_1 := \{z_1 \neq 0\}$ and $U_2 := \{z_2 \neq 0\}$ and let β', β'' be the braids we obtain by opening the first and second crossings of β, respectively. In particular, $U_1 = X_0(\beta' \cdot \Delta; w_0) \times \mathbb{C}^*$ and $U_2 = X_0(\beta'' \cdot \Delta; w_0) \times \mathbb{C}^*$. By the induction assumption, U_1 and U_2 can be covered up to codimension-2 by opening crossings in the positive braids β', β'', respectively. Moreover, the complement of $U_1 \cup U_2$ is $\{z_1 = 0\} \cap \{z_2 = 0\}$, which has codimension 2, and the required result follows. □

The toric charts $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta; w_0)$ used in Corollary 2.17 and Theorem 2.18 are constructed in Proposition 2.16, whose proof we now complete.

[4]Both lemmas are independent of the intervening material.

2.4. **Proof of Proposition 2.16.** Let us state and prove Lemma 2.22 and Lemma 2.23, which will conclude Proposition 2.16. For that, we use of the following notion:

Definition 2.19. Let T be the torus $(\mathbb{C}^*)^{n-1} = (\mathbb{C}^*)^n/\mathbb{C}^*$, and identify its character lattice with $\mathfrak{X}(T) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum a_i = 0\}$. Assume that T acts rationally on a domain R, and let $U = (u_{a,k}) \in M_n(R)$, $w \in S_n$. By definition, a matrix U is said to be *w-admissible* if $u_{a,k}$ is homogeneous of weight

$$\mathtt{wt}(u_{a,k}) = \mathbf{e}_{w(a)} - \mathbf{e}_{w(k)}$$

for every $a, k \leq n$. $\square$

Remark 2.20. (Characterization of w-admissibility) Note that we have three natural actions of the torus T on $M_n(R)$:

(1) Component-wise: $(\mathbf{t}.U)_{a,k} = \mathbf{t}.u_{a,k}$. (This is an algebra action.)
(2) By left multiplication: $(\mathbf{t}U)_{a,k} = t_a.u_{a,k}$.
(3) By right multiplication: $(U\mathbf{t})_{a,k} = t_k.u_{a,k}$.

Then, U is w-admissible if and only if $\mathbf{t}.U = \mathbf{t}^w U(\mathbf{t}^w)^{-1}$. $\square$

Before we proceed with Lemma 2.22, here is an example of a w-admissible matrix:

Example 2.21. *Let $w \in S_n$ be any permutation and assume z_0 is an invertible element of weight $\mathtt{wt}(z_0) = \mathbf{e}_{w(i+1)} - \mathbf{e}_{w(i)}$. Then the matrix $U_i(z_0) = Id + z_0^{-1} E_{i,i+1}$ is w-admissible.* $\square$

Consider a w-admissible matrix $U \in M_n(R)$ and z an element of weight $\mathtt{wt}(z) = \mathbf{e}_{w(a)} - \mathbf{e}_{w(b)} + \mathbf{e}_{w(m)} - \mathbf{e}_{w(k)}$ for some $a, b, m, k = 1, \dots, n$. Then the element $u_{a,k} + zu_{b,m}$ is homogeneous. The salient property of admissible matrices, which motivates their definition, is that they allow us to construct homogeneous elements for the torus action, as the following result shows.

Lemma 2.22. *Let $w \in S_n$ be a permutation, U^0 be an invertible upper-triangular w-admissible matrix, and $\beta = \sigma_{i_\ell} \cdots \sigma_{i_1}$ a positive braid word. Consider algebraically independent variables $z_\ell, \dots, z_1$ with weights*

$$\mathtt{wt}(z_k) = -\mathbf{e}_{w_{k-1}(i_k+1)} + \mathbf{e}_{w_{k-1}(i_k)}.$$

where $w_d = ws_{i_1} \cdots s_{i_d}$, and inductively define[5] the upper triangular matrices $U^1, \dots U^\ell$ and elements $z'_\ell, \dots, z'_1 \in R[z_\ell, \dots, z_1]$ by the equation

$$B_{i_{d+1}}(z_{d+1})U^d = U^{d+1} B_{i_{d+1}}(z'_{d+1}),$$

Then the following two facts holds:

(a) *The elements $z'_1, \dots, z'_{\ell+1}$ are all homogeneous with respect to the torus T-action and, moreover, $\mathtt{wt}(z'_d) = \mathtt{wt}(z_d)$ for every $d = 1, \dots, \ell$.*

(b) *For every $d = 0, \dots, \ell$, the matrix U^d is invertible, upper triangular, w_d-admissible and has entries in the polynomial ring $R[z_{d-1}, \dots, z_1]$.*

Proof. The matrices $U^0, \dots, U^\ell$ are readily invertible upper triangular. In order to prove the remaining claims, we induct on the length ℓ, with the base $\ell = 0$ holding by assumption.

For the inductive step, suppose that the statement holds for positive braids of length ℓ, and consider a positive braid $\beta = \sigma_{i_{\ell+1}}\sigma_{i_\ell} \cdots \sigma_{i_1}$ of length $\ell + 1$. Note that the matrices $U^0, U^1, \dots, U^\ell$ and the elements $z'_1, \dots, z'_\ell$ coincide with those for the braid $\sigma_{i_\ell} \cdots \sigma_{i_1}$, so we only need to show that the element $z'_{\ell+1}$ is homogeneous of the same weight as $z_{\ell+1}$, and that the matrix $U^{\ell+1}$ is $ws_{i_1} \cdots s_{i_{\ell+1}}$-admissible. To ease the notation, we will write $i := i_{\ell+1}$.

By the comment preceding the lemma, each of the entries of the matrix $B_i(z_{\ell+1})U^\ell$ is homogeneous. The $(i + 1, i + 1)$-entry of this matrix is $u^\ell_{i+1,i+1} z_{\ell+1} + u^\ell_{i,i+1}$. Dividing by $u^\ell_{i,i}$ we obtain that

$$z'_{\ell+1} = \frac{u^\ell_{i+1,i+1} z_{\ell+1} + u^\ell_{i,i+1}}{u^\ell_{i,i}}$$

[5] See Lemma 2.13.

is homogeneous. Since the diagonal entries of U^ℓ have weight 0 and every entry of U^ℓ is algebraically independent with $z_{\ell+1}$, we obtain that $z'_{\ell+1}$ is homogeneous of the same weight as $z_{\ell+1}$. Moreover, using again the w_ℓ-admissibility assumption for U^ℓ we have that every entry of the matrix $U^\ell B^{-1}(z'_{\ell+1})$ is homogeneous.

Now $U^{\ell+1} = B_i(z_{\ell+1})U^\ell B_i^{-1}(z'_{\ell+1})$. We check that this matrix is $w_{\ell+1} = w_\ell s_i$-admissible. Indeed, computing $(\mathbf{t}^{w_{\ell+1}})U^{\ell+1}(\mathbf{t}^{w_{\ell+1}})^{-1}$ we have

$$\begin{aligned}\mathbf{t}^{ws_i}B_i(z_{\ell+1})U^\ell B_i^{-1}(z'_{\ell+1})(\mathbf{t}^{ws_i})^{-1} &= (\mathbf{t}.B_i(z_{\ell+1}))\mathbf{t}^w U^\ell(\mathbf{t}^w)^{-1}(\mathbf{t}.B_i^{-1}(z'_{\ell+1}))\\ &= (\mathbf{t}.B_i(z_{\ell+1}))(\mathbf{t}.U^\ell)(\mathbf{t}.B_i^{-1}(z'_{\ell+1}))\\ &= \mathbf{t}.(U^{\ell+1})\end{aligned}$$

where the first equality follows from (2.4). This concludes the proof thanks to Remark 2.20. □

The proof of the following result is similar to that of Lemma 2.22 and left to the reader:

Lemma 2.23. *Let $U \in M_n(R)$ be a w-admissible upper-triangular matrix and $z_0 \in R$ homogeneous and invertible with weight $\mathtt{wt}(z_0) = -\mathbf{e}_{w(i+1)} + \mathbf{e}_{w(i)}$. Then the matrix $U' = D_i(z_0)UD_i^{-1}(z_0)$ is ws_i-admissible.* □

This concludes the necessary ingredients for Proposition 2.16, and thus completes our argument for Corollary 2.17 and Theorem 2.18. The following three subsections relate the results and constructions of Subsections 2.1, 2.2 and 2.3 to character varieties, through the work of A. Mellit [68], augmentation varieties, as featured in [53, 54], and open Bott-Samelson cells, according to [84, 86].

2.5. **Mellit's chart.** In this subsection we recast a construction by A. Mellit in the light of braid varieties, in particular defining a certain toric chart in $X_0(\beta\Delta, w_0)$, which we refer to as the Mellit chart. The main result of the subsection is that the Mellit chart can be obtained by our opening-crossing procedure from Subsection 2.3 above. In order to connect to [68], we need the following preliminary discussion.

Let $w \in S_n$ be a permutation and $C_w = \mathcal{B}w\mathcal{B} \subseteq \mathrm{GL}(n, \mathbb{C})$ the Bruhat cell corresponding to w, where $\mathcal{B} \subseteq \mathrm{GL}(n, \mathbb{C})$ is the Borel subgroup of upper-triangular matrices. Recall that the product of any two matrices in C_u and C_v belongs to C_{uv} if $\ell(uv) = \ell(u)+\ell(v)$. Consequently, for any reduced expression $u = s_{i_1}\cdots s_{i_\ell}$, the associated braid matrix $B_u(z_1, \ldots, z_\ell)$ belongs to the Bruhat cell C_u.[6]

Proposition 2.24. *Let $u = s_{i_1}\cdots s_{i_\ell}$ be a reduced expression and suppose that $\ell(us_i) = \ell(u) - 1$. Then there exists $k \in \mathbb{N}$ such that:*

(a) *The matrix $B_u(z_1, \ldots, z_\ell)B_i(z)$ belongs to the Bruhat cell C_u if and only if $z_k \neq 0$,*

(b) *In case $z_k \neq 0$, we can uniquely write $B_u(z_1, \ldots, z_\ell)B_i(z) = UB_u(z'_1, \ldots, z'_\ell)$ for a certain upper-triangular matrix U.*

Proof. Since $\ell(us_i) = \ell(u) - 1$, there exists $k \in \mathbb{N}$ such that $us_i = s_{i_1}\cdots \widehat{s_{i_k}}\cdots s_{i_\ell}$.[7] That is, we can write $u = u_1 s_{i_k} u_2$ such that $s_{i_k}u_2 = u_2 s_i$, and thus $us_i = u_1 s_{i_k} u_2 s_i = u_1 s_{i_k} s_{i_k} u_2 = u_1 u_2$. This implies the following equation for the braid matrices:

$$B_u(z_1, \ldots, z_\ell)B_i(z) = B_{u_1}(z_1, \ldots, z_{k-1})B_{i_k}(z_k)B_{i_k}(z')B_{u_2}(z'_{k+1}, \ldots, z'_\ell),$$

where $z', z'_{k+1}, \ldots, z'_\ell$ are some functions of $z, z_{k+1}, \ldots, z_\ell$. If $z_k \neq 0$, then we can further write $B_{i_k}(z_k)B_{i_k}(z') = UB_{i_k}(z'')$, so

$$B_u(z_1, \ldots, z_\ell)B_i(z) = \widetilde{U}B_{u_1}(z'_1, \ldots, z'_{k-1})B_{i_k}(z'')B_{u_2}(z'_{k+1}, \ldots, z'_\ell) = \widetilde{U}B_u(z'_1, \ldots, z'_{k-1}, z'', z'_{k+1}, \ldots, z'_\ell)$$

and the result is in the Bruhat cell C_u. If instead $z_k = 0$, then $B_{i_k}(z_k)B_{i_k}(z')$ is upper-triangular, and $B_u(z_1, \ldots, z_\ell)B_i(z)$ is in the Bruhat cell $C_{u_1u_2}$, which is disjoint from C_u. □

[6]Recall that we interchangeably use the notation s_i and σ_i for the Artin generators of the braid group, which is particularly well-suited when comparing to the notation used in [68].

[7]This is known as *exchange property* for the Coxeter group S_n.

Example 2.25. *Consider $\beta_1 = s_1s_2s_1s_1$ and $\beta_2 = s_1s_2s_1s_2$. Then the braid matrix $B_{\beta_1}(z_1, z_2, z_3, z_4)$ is in the Bruhat cell $C_{s_1s_2s_1}$ if and only if $z_3 \neq 0$. In contrast, the braid matrix $B_{\beta_2}(z_1, z_2, z_3, z_4)$ is in the Bruhat cell $C_{s_1s_2s_1}$ if and only if $z_1 \neq 0$. In both cases the reduced expression is $u = s_1s_2s_1$.* □

Remark 2.26. The index $k \in \mathbb{N}$ from Proposition 2.24 is unique and can be described geometrically, as follows. Draw a braid diagram for u, labeling the strands 1 to n on the right. Since $\ell(us_i) = \ell(u)-1$, the i-th and $(i+1)$-st strands intersect somewhere in the diagram for u. Given that u is reduced, they intersect exactly once. The index k corresponds to this intersection point. □

Let us now compare our construction to A. Mellit's [68], with $\beta\Delta = s_{i_1} \cdots s_{i_{\ell+\binom{n}{2}}}$ a positive braid. In [68, Section 5.4], A. Mellit defines a sequence of permutations $p_0 = 1, p_1, \ldots, p_{\ell+\binom{n}{2}}$ by the rules:

(a) If $\ell(p_{k-1}s_{i_k}) = \ell(p_{k-1}) + 1$ then $p_k = p_{k-1}s_{i_k}$,
(b) If $\ell(p_{k-1}s_{i_k}) = \ell(p_{k-1}) - 1$ then $p_k = p_{k-1}$.

In his terminology, this sequence is a *walk which never goes down.* Let us now describe the toric chart used in [68]:

Definition 2.27 (Mellit Chart)**.** Let β be a positive n-braid word, the Mellit chart $\mathfrak{M} \subseteq X_0(\beta\Delta, w_0)$ is defined as the locus of $z_1, \ldots, z_s$ such that

$$B_{i_1}(z_1) \cdots B_{i_s}(z_s) \in C_{p_s} \text{ for all } s \leq \ell + \binom{n}{2}. \tag{2.7}$$

Note that $\mathfrak{M} \subseteq X_0(\beta\Delta, w_0)$ is codimension-0 and Zariski open in $X_0(\beta\Delta, w_0)$. □

At this stage, our Corollary 2.17 provides many toric charts $T_{\tau(\beta)}$ for $X_0(\beta\Delta, w_0)$, (surjectively) indexed by orderings $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings. The toric chart $\mathfrak{M}$ introduced in Definition 2.27 is also a subset of $X_0(\beta\Delta, w_0)$, and it is thus natural to ask whether $\mathfrak{M}$ is of the form $T_{\tau(\beta)}$ and, if so, for which ordering $\tau(\beta)$ this is the case. This is answered in our next result (and its proof):

Theorem 2.28. *Let β be a positive braid word. Then there exists an ordering $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings such that $T_{\tau(\beta)} \subseteq X_0(\beta\Delta, w_0)$ coincides with the Mellit chart $\mathfrak{M} \subseteq X_0(\beta\Delta, w_0)$.*

Proof. The ordering $\tau(\beta)$ in which we open the crossings is as follows. First, we find the smallest j such that $p_{j-1} = p_j$. This means that $p_{j-1} = s_{i_1} \cdots s_{i_{j-1}}$ is a reduced word and $\ell(p_{j-1}s_{i_j}) = \ell(p_{j-1}) - 1$. The condition (2.7) holds automatically for $s < j$, and for $s = j$ we can apply Proposition 2.24: there exists some $k < j$ such that $B_{i_1}(z_1) \cdots B_{i_j}(z_j) \in C_{p_j}$ if and only if $z_k \neq 0$.

It follows from Remark 2.26 that the crossing with index k is in the braid β, and never in Δ. We can open this crossing and obtain a new braid $\beta'\Delta$. By Proposition 2.24, a point in $X_0(\beta\Delta, w_0)$ is in the Mellit chart if and only if the corresponding point in $X_0(\beta'\Delta, w_0)$ is in the respective chart. This process can be continued iteratively. Eventually, all crossings in β will be exhausted, and we reach a reduced expression Δ, which satisfies the defining inclusion (2.7) automatically. □

Example 2.29. *Consider the positive 3-braid $\beta = \sigma_1\sigma_2\sigma_1$, and thus $\beta\Delta = \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1$. By opening the third crossing σ_1, from the left, we reach the braid word $\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$. Then we open the first (leftmost) σ_1 crossing and obtain* 2121*. Finally, opening again the first (leftmost) crossing σ_1 in the resulting braid (which corresponds to the second crossing in the original braid) we reach the positive braid word $\Delta = \sigma_1\sigma_2\sigma_1$. This sequence of crossings $\tau(\beta)$ yields a toric chart $T_{\tau(\beta)} \subseteq X_0(\beta\Delta, w_0)$ which coincides with the Mellit chart $\mathfrak{M} \subseteq X_0(\beta\Delta, w_0)$.* □

This concludes our discussion on the Mellit chart and the relation between our Corollary 2.17 and [68, Section 5]. Let us shift our focus towards an class of algebraic varieties which are central to the study of Legendrian links in contact 3-manifolds, augmentation varieties.

2.6. **Augmentation varieties as quotient braid varieties.** In this subsection, we establish a connection between braid varieties and augmentation varieties. The latter are a class of varieties that feature saliently in the study of Floer-theoretic invariants associated to Legendrian links $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$. The reader is referred to [38] for the basics of 3-dimensional contact topology, [70] for an excellent survey on Floer-theoretic invariants of Legendrian knots, and [23, 53, 54] for further details.

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word, and $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ the Legendrian links associated to the (-1)-framed closure of the braid β, see [16, 20]. Let us also choose a collection of marked (base) points $\mathfrak{t} \subseteq \beta$ on the Legendrian links, following [71, 72]. In our case, the two choices for marked points that we use are: a choice of *one* marked point per strand of the braid β, this collection of marked points will be denoted by $\mathfrak{t}_s$; a choice of *one* marked point per component of the Legendrian link $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, this collection will be denoted by $\mathfrak{t}_c$. By convention, we place any marked points on the strands and to the right of all crossings in β.[8]

Let $\mathcal{A}(\beta, \mathfrak{t})$ be the Legendrian Contact DGA of the Legendrian link $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, also known as the Chekanov-Eliashberg DGA. This differential graded algebra (DGA) is an invariant of the Legendrian link $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ (with marked points) up to Legendrian isotopy. It was defined by Y. Chekanov [23] over $\mathbb{Z}_2$-coefficients, and latter lifted to $\mathbb{Z}$-coefficients and marked points [71, 72], see [70] for a survey. The differential $\mathcal{A}(\beta, \mathfrak{t})$ is given by a Floer-theoretical count of (pseudo)holomorphic strips whose asymptotics are governed by the Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$. By definition, the *augmentation variety* $\mathrm{Aug}(\beta, \mathfrak{t})$ associated to $(\beta, \mathfrak{t})$ is the affine algebraic variety $\mathrm{Aug}(\beta, \mathfrak{t}) := \mathrm{Hom}_{\mathrm{dg}}(\mathcal{A}(\beta, \mathfrak{t}), \mathbb{C})$ of all DGA morphisms from $\mathcal{A}(\beta, \mathfrak{t})$ to the DGA $\mathbb{C}$, the latter being concentrated in degree 0 and with trivial differential.

In the case of Legendrian links $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ associated to positive braids, $\Lambda \simeq \Lambda(\beta)$, augmentation varieties $\mathrm{Aug}(\beta, \mathfrak{t})$ are closely related braid varieties. This will follow rather simply from the work of T. Kálmán [54], and it is the content of the following:

Theorem 2.30. *Let β be a positive braid word, $[\beta] \in \mathrm{Br}_n^+$. The following two statements hold:*

(i) *There exists an algebraic isomorphism* $\mathrm{Aug}(\beta, \mathfrak{t}_s) \cong X_0(\beta \cdot \Delta; w_0)$.

(ii) *Let $T \subseteq (\mathbb{C}^*)^n$ be the algebraic torus determined by $t_{w^{-1}(i)} = 1$ if the ith strand of the braid β has a marked point in $\mathfrak{t}_c$. Then, there exists an algebraic isomorphism*
$$\mathrm{Aug}(\beta, \mathfrak{t}_c) \cong X_0(\beta \cdot \Delta; w_0)/T.$$

Proof. Let us use the following characterization by T. Kálmán [53, 54] (see also [20]): if β is a positive braid word and $i_1, \ldots, i_s$ are strands that carry a marked point (to the right of every crossing) then the augmentation variety is the affine subvariety of $\mathbb{C}^{\ell(\beta)+\binom{n}{2}} \times (\mathbb{C}^*)^s$ given by the equation

$$B_\beta(z) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ c_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & 1 \end{pmatrix} \mathrm{diag}(t_1, \ldots, t_n) \text{ is upper triangular with a prescribed diagonal} \tag{2.8}$$

where the notation follows the convention that in $\mathrm{diag}(t_1, \ldots, t_n)$ we have $t_i = 1$ if $i \neq i_1, \ldots, i_s$. For the choice of marked points $\mathfrak{t}_s$, this reduces to $B_\beta(z)B_\Delta(u)w_0$ being upper triangular, which is precisely the definition of $X_0(\beta \cdot \Delta; w_0)$. This establishes the statement in (i). For the choice of marked points $\mathfrak{t}_c$, as in (ii), Equation (2.8) reduces to $B_\beta(z)B_\Delta(u)w_0$ being upper triangular with a prescribed diagonal outside of the strands carrying marked points. Since the action of T on $X_0(\beta \cdot \Delta; w_0)$ is free (see Remark 2.12) the quotient map $X_0(\beta \cdot \Delta; w_0) \to X_0(\beta \cdot \Delta; w_0)/T$ is a principal T-bundle. In consequence, $X_0(\beta \cdot \Delta; w_0)/T$ is equivalent to the closed subvariety of $X_0(\beta \cdot \Delta; w_0)$ given by prescribing the diagonal elements in $B_{\beta \cdot \Delta} w_0$ at entries corresponding to strands not carrying marked points. $\square$

In contact geometry, opening a crossing from $\beta = \beta_1 \sigma_i \beta_2$ to $\beta' = \beta_1 \beta_2$ can be realized[9] by an embedded exact Lagrangian cobordism $L_i \subseteq (\mathbb{R}^3 \times \mathbb{R}_t, d(e^t \alpha))$ in the symplectization of $(\mathbb{R}^3, \xi_{\mathrm{st}})$, with $\partial L_i = \partial_- L_i \cup \partial_+ L_i$ and $\partial_- L_i = \Lambda(\beta')$ and $\partial_+ L_i = \Lambda(\beta)$ [1, 10]. It follows from the Floer-theoretic functoriality proven in [27, 75] that such a Lagrangian cobordism induces a map Φ_{L_i} : $\mathrm{Aug}(\beta', \mathfrak{t}) \longrightarrow \mathrm{Aug}(\beta, \mathfrak{t})$ between augmentation varieties. It follows from [20, 27] that the ($\mathbb{Z}$-lifted)

[8] Though not essential, this convention will be useful in simplifying some statements. Note also that $\mathfrak{t}_c$ technically depends on a choice of strand per component of $\Lambda(\beta)$, but for the sake of readability we prefer to not include this into our notation.

[9] This is correct in the case that the positive braid has a half-twist remaining [20, 27], which will always be the case in our context.

Floer-theoretical map Φ_{L_i} agrees with the (quotient of the) map Ω_{σ_i} we constructed in Subsection 2.3.

The toric charts we constructed in Corollary 5.5, using Proposition 2.16, can now be used to stratify the augmentation varieties in Theorem 2.30, as follows:

Corollary 2.31. *Let β be a positive braid word, $[\beta] \in \mathrm{Br}_n^+$, with $c(\beta) = k$.*[10] *For each ordering $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings of β there exists codimension-0 toric charts $T^c_{\tau(\beta)} \subseteq \mathrm{Aug}(\beta, \mathfrak{t}_c)$ and $T^s_{\tau(\beta)} \subseteq \mathrm{Aug}(\beta, \mathfrak{t}_s)$, with $T^c_{\tau(\beta)} \cong (\mathbb{C}^*)^{\ell(\beta)-n+k}$ and $T^s_{\tau(\beta)} \cong (\mathbb{C}^*)^{\ell(\beta)}$. In both cases, the complements*

$$\mathrm{Aug}(\beta, \mathfrak{t}_c) \setminus \left(\bigcup_{\tau(\beta) \in S_{\ell(\beta)}} T^c_{\tau(\beta)} \right) \subseteq \mathrm{Aug}(\beta, \mathfrak{t}_c), \qquad \mathrm{Aug}(\beta, \mathfrak{t}_s) \setminus \left(\bigcup_{\tau(\beta) \in S_{\ell(\beta)}} T^s_{\tau(\beta)} \right) \subseteq \mathrm{Aug}(\beta, \mathfrak{t}_s)$$

have codimension at least 2.

Proof. In view of Theorem 2.30, only the statements for $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ remain unproven, but these follow from the T-stability of the toric charts on the braid variety $X_0(\beta \cdot \Delta; w_0)$, cf. Corollary 2.17. □

2.7. **Open Bott-Samelson varieties.** This section is not required for the rest of the manuscript: it is provided here for contextual completeness with respect to the articles [16, 37, 84, 85]. The purpose of this section is to relate the braid variety $X_0(\beta)$ to the (diagonal) open Bott-Samelson variety $\mathrm{OBS}(\beta)$ associated to the braid β. This is achieved in Theorem 2.34 below, after a brief reminder on Bott-Samelson varieties.

Consider $G := \mathrm{GL}(n, \mathbb{C})$, $\mathcal{B} \subseteq G$ the Borel subgroup of upper-triangular matrices and the flag variety $\mathcal{F}\ell := G/\mathcal{B}$. The projective variety $\mathcal{F}\ell$ is the moduli space of complete flags of subspaces in $\mathbb{C}^n$: an element $\mathcal{F} \in \mathcal{F}\ell$ is a flag $\mathcal{F} = (F_1 \subseteq \cdots \subseteq F_n)$ where $\dim F_i = i$. Given a flag $\mathcal{F} \in \mathcal{F}\ell$, we can choose a basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that $F_j = \langle v_1, \ldots, v_j \rangle$ for $j = 1, \ldots, n$; we denote by $V_\mathcal{F} \in G$ the matrix whose columns are the vectors v_i expressed in the standard basis. Conversely, given a matrix $V \in G$, we can consider a flag $\mathcal{F}^V = (F_1 \subseteq \cdots \subseteq F_n)$ where F_j is the span of the first j columns of the matrix V. In this correspondence, two flags are equal $\mathcal{F}^V = \mathcal{F}^{V'}$ if and only if their matrices V, V' are related by an upper triangular matrix, i.e. $V = V'U$ for some $U \in \mathcal{B}$.

By definition, two flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}\ell$ are in relative position s_i, $i \in [1, n-1]$, if $F_j = F'_j$ for $j \neq i$ and $F_i \neq F'_i$. In terms their matrices, the flags $\mathcal{F}^V, \mathcal{F}^{V'}$ are in relative position s_i if and only if there exist upper-triangular matrices A_1 and A_2 such that $V' = VA_1 s_i A_2$, where s_i is understood as a permutation matrix.

Remark 2.32. Since the permutation matrix $s_i = B_i(0)$ is a braid matrix with the variable set to zero, it follows from Lemma 2.13 that the flags $\mathcal{F}^V$ and $\mathcal{F}^{V'}$ are in relative position π_i if and only if there exist an upper-triangular matrix U and $z \in \mathbb{C}$ such that $V' = VUB_i(z)$. □

Building on the articles [11, 25], and the subsequent developments [16, 84, 85, 86], we introduce the two algebraic varieties $\mathrm{OBS}(\beta)$ and $\mathrm{OBS}'(\beta)$, as follows:

Definition 2.33. Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ be a positive braid word.

(i) The open Bott-Samelson variety $\mathrm{OBS}(\beta) \subseteq \mathcal{F}\ell^{\ell+1}$ associated to β is the moduli space of $(\ell+1)$-tuples of flags $(\mathcal{F}_0, \ldots, \mathcal{F}_\ell)$ such that consecutive flags $\mathcal{F}_{k-1}, \mathcal{F}_k$ are in relative position s_{i_k}, for each $k \in [1, \ell]$.

(ii) The *diagonal* open Bott-Samelson variety $\mathrm{OBS}'(\beta) \subseteq \mathrm{OBS}(\beta)$ is the closed subvariety defined by the additional condition that $\mathcal{F}_0 = \mathcal{F}_\ell$. □

The diagonal open Bott-Samelson variety $\mathrm{OBS}'(\beta)$ will be related to the braid variety, as we now explain. First, let us construct a map $\pi : G \times X_0(\beta) \to \mathrm{OBS}'(\beta)$ as follows. Consider a point $(z_1, \ldots, z_\ell) \in X_0(\beta)$ and a matrix $V \in G$, and define $V_k := VB_{i_1}(z_1) \cdots B_{i_k}(z_k) \in G$. The map π is then defined by:

$$\pi : G \times X_0(\beta) \longrightarrow \mathrm{OBS}'(\beta), \quad \pi(V, z_1, \ldots, z_\ell) := (\mathcal{F}^V, \mathcal{F}^{V_1}, \ldots, \mathcal{F}^{V_\ell}).$$

[10]Geometrically, the Legendrian link has $\Lambda(\beta)$ has k connected components.

It follows from Remark 2.32 that $\pi(V, z_1, \ldots, z_\ell) \in \mathrm{OBS}(\beta)$ and since $V_\ell = VB_\beta(z_1, \ldots, z_\ell)$, and $(z_1, \ldots, z_\ell) \in X_0(\beta)$, we actually have that $\pi(V, z_1, \ldots, z_\ell) \in \mathrm{OBS}'(\beta)$. Thus the image of π belongs to $\mathrm{OBS}'(\beta) \subseteq \mathrm{OBS}(\beta)$, as written above. This map is, in general, not an isomorphism. Nevertheless, we will now construct a right $\mathcal{B}$-action on the product $G \times X_0(\beta)$, and π will descend to an isomorphism on the quotient.

Indeed, consider an upper-triangular matrix $U = U^0 \in \mathcal{B}$ and define $z'_1, \ldots, z'_\ell$ and $U^1, \ldots, U^\ell \in \mathcal{B}$ inductively via the equation

$$B_{i_{\ell-k}}(z_{\ell-k})U^k = U^{k+1}B_{i_{\ell-k}}(z'_{\ell-k}). \tag{2.9}$$

It follows from the equation $U^\ell B_\beta(z_1, \ldots, z_\ell) = B_\beta(z'_1, \ldots, z'_\ell)U^0$ that $(z_1, \ldots, z_\ell) \in X_0(\beta)$ if and only if $(z'_1, \ldots, z'_\ell) \in X_0(\beta)$. For each $(V, z_1, \ldots, z_\ell) \in G \times X_0(\beta)$ and upper-triangular matrix $U = U^0 \in \mathcal{B}$, we define its (right) action by:

$$(V, z_1, \ldots, z_\ell) \cdot U := (VU^\ell, z'_1, \ldots, z'_\ell).$$

The usefulness of this right action is manifest in the main result of this subsection:

Theorem 2.34. *Let β be a positive braid word, $G = \mathrm{GL}(n, \mathbb{C})$ and $\mathcal{B} \subseteq G$ the Borel subgroup of upper-triangular matrices. Then*

(i) *The right $\mathcal{B}$-action on $G \times X_0(\beta)$ defined above is free.*

(ii) *The map $\pi : G \times X_0(\beta) \longrightarrow \mathrm{OBS}'(\beta)$ induces an isomorphism $(G \times X_0(\beta))/\mathcal{B} \cong \mathrm{OBS}'(\beta)$.*

Proof. Let us first prove the freeness of the right $\mathcal{B}$-action. Indeed, suppose that there exists a fixed point, i.e. there exist $U \in \mathcal{B}$ and $(V, z_1, \ldots, z_\ell) \in G \times X_0(\beta)$ such that

$$(V, z_1, \ldots, z_\ell) \cdot U = (V, z_1, \ldots, z_\ell).$$

Since $z'_j = z_j$ for every $j \in [1, \ell]$, it follows from Equation (2.9) that the matrices $U, U^1, \ldots, U^\ell$ are pair-wise conjugate. In particular, the initial upper-triangular matrix U is conjugate to U^ℓ. Nevertheless, the condition $VU^\ell = V$ implies that $U^\ell = \mathrm{Id}$, and thus no fixed point may exist. The action is thus free.

Second, let us show that the map π is surjective, onto the diagonal open Bott-Samelson variety $\mathrm{OBS}'(\beta)$. For that, consider a point $(\mathcal{F}_0, \ldots, \mathcal{F}_\ell) \in \mathrm{OBS}'(\beta)$ and let $V \in G$ be any matrix such that $\mathcal{F}^V = \mathcal{F}_0$. Thanks to Remark 2.32, we have that there exist upper-triangular matrices $U^1, \ldots, U^\ell$ and $z_1, \ldots, z_\ell \in \mathbb{C}$ such that

$$\mathcal{F}_k = \mathcal{F}^{VU_1B_{i_1}(z_1)\cdots U_kB_{i_k}(z_k)}.$$

Now, use Lemma 2.13 to slide all the upper triangular matrices $U_2, \ldots, U_\ell$ to the right; this yields obtain upper-triangular matrices $U'_1 = U_1, U'_2, \ldots, U'_\ell$ and $\hat{z}_1, \ldots, \hat{z'}_\ell$ with the property that

$$VU'_1 \cdots U'_\ell B_{i_1}(\hat{z}_1) \cdots B_{i_k}(\hat{z}_k) = VU_1B_{i_1}(\hat{z}_1) \cdots U_kB_{i_k}(\hat{z}_k)\hat{U},$$

where $\hat{U}$ is an upper-triangular matrix depending on k. This implies that $\pi(VU'_1 \cdots U'_\ell, \hat{z}_1, \ldots, \hat{z}_\ell) = (\mathcal{F}_0, \ldots, \mathcal{F}_\ell)$. It remains to show that $(\hat{z}_1, \ldots, \hat{z}_\ell) \in X_0(\beta)$, that is, the matrix $B_\beta(\hat{z}_1, \ldots, \hat{z}_\ell)$ is upper-triangular. Since $\mathcal{F}_0 = \mathcal{F}_\ell$, the matrices V and $VU'_1 \cdots U'_\ell B_\ell(\hat{z}_1, \ldots, \hat{z}_\ell)$ differ by an upper-triangular matrix. Since $U'_1, \ldots, U'_\ell$ are upper-triangular, the result follows and π is surjective.

Third, let us prove that the map π is $\mathcal{B}$-invariant; we need to check that for every k the matrices $VB_{i_1}(z_1) \cdots B_{i_k}(z_k)$ and $VU^\ell B_{i_1}(z'_1) \cdots B_{i_k}(z'_k)$ differ by an upper-triangular matrix. That said, it follows from Equation (2.9) that this matrix is precisely $U^{\ell-k}$, which proves $\mathcal{B}$-invariance.

Finally, we must show that if $\pi(V, z_1, \ldots, z_\ell) = \pi(V', z'_1, \ldots, z'_\ell)$ then there exist an upper triangular matrix U such that $(V', z'_1, \ldots, z'_\ell) = (V, z_1, \ldots, z_\ell) \cdot U$. For that, note that $\mathcal{F}^V = \mathcal{F}^{V'}$ implies that there exist an upper-triangular matrix, say U^ℓ, such that $V' = VU^\ell$. Since $\mathcal{F}^{VB_{i_1}(z_1)} = \mathcal{F}^{V'B'_{i_1}(z'_1)}$, there also exist an upper-triangular matrix, say $U^{\ell-1}$ such that $V'B'_{i_1}(z'_1) = VB_{i_1}(z_1)U^{\ell-1}$. In consequence, we obtain the equality $VU^\ell B_{i_1}(z'_1) = VB_{i_1}(z_1)U^{\ell-1}$, and thus $U^\ell B_{i_1}(z'_1) = B_{i_1}(z_1)U^{\ell-1}$. Note that this is precisely Equation (2.9). We iterate this procedure until we find U^0, which is the required upper-triangular matrix. This concludes the proof of the statement. $\square$

The concludes our discussion relating braid varieties to open Bott-Samelson varieties. Let us now move to the construction of a holomorphic symplectic structure on braid varieties $X_0(\beta)$.

Remark 2.35. As a side note, the homotopy types of the varieties $X_0(\beta)$ and $\mathrm{OBS}'(\beta)$ appear to be related to the spectra constructed in [58]. This remains to be explored. □

3. Holomorphic Symplectic Structure

This section constructs a holomorphic symplectic structure on the (quotient) braid varieties $X_0(\beta\Delta; w_0)/T$, establishing the remainder of Theorem 1.1.(iii). In particular, Theorem 2.30 will imply that the augmentation variety associated to a Legendrian link $\Lambda(\beta)$, β a positive braid (word), is holomorphic symplectic. In addition, the toric charts we built in Corollary 2.17 will actually be (exponential) Darboux charts for this holomorphic symplectic structure. The construction we present draws from the literature on character varieties, where the holomorphic symplectic structures on character varieties have a central role, starting with the Atiyah-Bott-Goldman structures [3, 39] and continuing with, e.g., the work of P. Boalch and L. Jeffrey [6, 7, 52, 68].

3.1. Construction of a 2-form.

First, let us review the construction of a 2-form on the braid variety $X_0(\beta)$ according to [52, 68], often referred to as the tautological 2-form. For that, let $\theta := f^{-1}df$ and $\theta^R := df f^{-1}$ denote respectively the left- and right-invariant algebraic 1-forms on the (complex) Lie group $G = \mathrm{GL}(n, \mathbb{C})$; these 1-forms are valued in the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$, and θ is referred to as the Maurer-Cartan form. The following two facts can be readily verified:

(a) The 3-form $\Omega := \mathrm{Tr}(\theta[\theta, \theta])$ is closed and represents a class in $H^3(G)$. This class is in fact dual to the fundamental class of (a copy of) $\mathbb{S}^3 \simeq SU(2) \subset GL(n, \mathbb{C})$.

(b) There is a 2-form $(f|g) := \mathrm{Tr}(\pi_1^*\theta \wedge \pi_2^*\theta^R) = \mathrm{Tr}(f^{-1}df \wedge dgg^{-1})$ on $G \times G$ satisfying the following two "cocycle conditions":

$$d(f|g) = \pi_1^*\Omega - m^*\Omega + \pi_2^*\Omega, \tag{3.1}$$

$$(g|h) - (fg|h) + (f|gh) - (f|g) = 0, \tag{3.2}$$

where $\pi_1, \pi_2, m : G \times G \to G$ are the two projections and the Lie group multiplication map.

Let X be an arbitrary algebraic variety. By definition, a map $f : X \to G$ is Ω-trivial if $f^*\Omega = d\omega$ for some 2-form ω on X. Now, suppose that two maps $f : X \to G$ and $g : Y \to G$ are Ω-trivial, then the product

$$f \cdot g : X \times Y \xrightarrow{f\times g} G \times G \xrightarrow{m} G$$

is also Ω-trivial. Indeed, if $f^*\Omega = d\omega_X$ and $g^*\Omega = d\omega_Y$ then (3.1) implies

$$(f \cdot g)^*\Omega = d(\omega_X + \omega_Y + (f|g)).$$

In general, if $f_i : X_i \to G, i \in [1, r]$ are Ω-trivial maps then we can define the form

$$\omega = \sum \omega_{X_i} + (f_1|f_2) + (f_1 f_2|f_3) + \ldots + (f_1 \cdots f_{r-1}|f_r) =$$

$$\sum \omega_{X_i} + (f_1|f_2| \cdots |f_r).$$

The condition (3.2) implies that this operation defines an associative convolution $(f_1|f_2| \cdots |f_r)$ on collections of Ω-trivial maps. The following identity will be useful for us, and follows immediately from the definition:

$$(f_1| \cdots |f_r) = (f_1| \cdots |f_j f_{j+1}| \cdots |f_r) + (f_j|f_{j+1}). \tag{3.3}$$

This summarizes the necessary ingredients. Consider a positive braid word β, with length $r = \ell(\beta)$ and let us apply this construction to our braid varieties $X_0(\beta)$, as follows. The first key fact is that the maps $B_i(z) : \mathbb{C} \to G$ given by the braid matrices are Ω-trivial with vanishing primitive 2-form $\omega = 0$. Hence, the map $B_\beta : \mathbb{C}^r \to G$ is also Ω-trivial with primitive being the 2-form

$$\omega_\beta = (B_{i_1}(z_1)|B_{i_2}(z_2)) + (B_{i_1}(z_1)B_{i_2}(z_2)|B_{i_3}(z_3)) + \ldots + (B_{i_1}(z_1) \cdots B_{i_{k-1}}(z_{k-1})|B_{i_k}(z_k)). \tag{3.4}$$

Lemma 3.1. *The restriction of the 2-form ω_β to the braid variety $X_0(\beta)$ is closed.*

Proof. Note that the map $B_\beta : X_0(\beta) \longrightarrow G$ lands in the subgroup of upper-triangular matrices, and the restriction of the 3-form Ω to the space of upper-triangular matrices vanishes. Therefore, since d commutes with pull-back, we have

$$d\omega_\beta = B_\beta^*\Omega = 0,$$

i.e. ω_β is a closed 2-form. $\square$

The following example will prove useful:

Example 3.2. *Let $\Delta \in \mathrm{Br}_n^+$ be the positive braid (word) associated to the half-twist; then we have that the 2-form $\omega_\Delta = 0$ vanishes. (Thus, it is closed.) Following Lemma 2.3, we can write*

$$B_{\Delta^2} = B_\Delta(c)B_\Delta(u) = Lw_0 \cdot w_0U = LU,$$

where two copies of B_Δ depend on two sets of independent variables c_{ij} and u_{ij}. The 2-form associated to Δ^2 then reads:

$$\omega_{\Delta^2} = \omega_\Delta(c) + (B_\Delta(c)|B_\Delta(u)) + \omega_\Delta(u) = (B_\Delta(c)|B_\Delta(u)) =$$
$$(Lw_0|w_0U) = (L|w_0|w_0|U) = (L|w_0w_0|U) = (L|U).$$

$\square$

Consider now the toric charts $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta^2)$ constructed in Corollary 2.17.

Lemma 3.3. *Let β be a positive braid (word) and $\tau(\beta) \in S_{\ell(\beta)}$. The restriction of 2-form $\omega_{\beta\cdot\Delta^2}$ to any toric chart $T_{\tau(\beta)} \subseteq X_0(\beta \cdot \Delta^2)$ has constant coefficients in the canonical coordinates (associated to the D_i matrices).*

Proof. By Lemma 2.3, we can write

$$B_{\beta\cdot\Delta^2} = B_\beta B_{\Delta^2} = B_{i_1}(z_1)\cdots B_{i_r}(z_r)LU.$$

By Example 3.2, we can also write

$$\omega_{\beta\cdot\Delta^2} = \omega_\beta + (B_\beta|B_{\Delta^2}) + \omega_{\Delta^2} =$$
$$\omega_\beta + (B_\beta|LU) + (L|U) = (B_{i_1}(z_1)|\cdots|B_{i_r}(z_r)|L|U).$$

Next, we need to understand the behavior of the 2-form under opening the crossings according to $\tau(\beta)$, as this determines the construction of the toric chart $T_{\tau(\beta)}$. By using the decomposition in Equation 2.6, we can write

$$(\cdots|B_{i_s}(z_s)|\cdots) =$$
$$(\cdots|U_{i_s}(z_s)|D_i(z_s)|L_i(z_s)|\cdots) - (U_{i_s}(z_s)|D_i(z_s)|L_i(z_s)),$$

and clearly $(U_{i_s}(z_s)|D_i(z_s)|L_i(z_s)) = 0$. Next, assume that U is an upper uni-triangular matrix, then

$$(\cdots|B_i(z)|U|\cdots) = (\cdots|B_i(z)U|\cdots) + (B_i(z)|U) =$$
$$(\cdots|\widetilde{U}B_i(z')|\cdots) + (B_i(z)|U) = (\cdots|\widetilde{U}|B_i(z')|\cdots) + (B_i(z)|U) - (\widetilde{U}|B_i(z')).$$

The terms $(B_i(z)|U), (\widetilde{U}|B_i(z'))$ in fact vanish. Indeed, observe that

$$B_i^{-1}(z)dB_i(z) = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & dz \end{pmatrix} = \begin{pmatrix} 0 & dz \\ 0 & 0 \end{pmatrix},$$

while $dU \cdot U^{-1}$ is upper-triangular, so $(B_i(z)|U) = 0$. Similarly,

$$dB_i(z') \cdot B_i(z')^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & dz' \end{pmatrix}\begin{pmatrix} -z' & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dz' & 0 \end{pmatrix},$$

while $\widetilde{U}_{i,i+1} = 0$ and hence $(\widetilde{U}|B_i(z')) = 0$. We conclude that

$$(\cdots|B_i(z)|U|\cdots) = (\cdots|\widetilde{U}|B_i(z')|\cdots),$$

and similarly $(\cdots|D_i(z)|U|\cdots) = (\cdots|\widetilde{U}|D_i(z)|\cdots)$. The conclusion from this computation is that the 2-form $\omega_{\beta\cdot\Delta^2}$ does not change as we move U to the left. (Similarly, it does not change as we move lower-triangular matrices to the right.) In conclusion, we are left with several upper uni-triangular matrices, followed by several diagonal matrices and by several lower uni-triangular matrices. Since

$(U|U') = (L|L')$ for any pair of upper (resp. lower) uni-triangular matrices, we can consolidate all upper and all lower uni-triangular matrices and write

$$\omega_{\beta\cdot\Delta^2} = (\widetilde{U}|D_1|\cdots|D_k|\widetilde{L}|U).$$

Since $\widetilde{U}D_1\cdots D_k\widetilde{L}U$ is upper-triangular, we get $\widetilde{L} = 1$. Thus

$$\omega_{\beta\cdot\Delta^2} = (\widetilde{U}|D_1|\cdots|D_k|U) = (D_1|\cdots|D_k) = \sum_{i<j} \mathrm{Tr}(d\log(D_i) \wedge d\log(D_j)).$$

and the result follows. □

The proof of Lemma 3.3 actually shows that the entries of the D_i matrices associated to (opening the crossings for) β are quite close to (exponential) Darboux-like coordinates[11], only differing from an exponential Darboux-like chart by a linear change in constant coefficients. We have now discussed closedness of the 2-form $\omega_{\beta\cdot\Delta^2}$ and its expression in the toric charts $T_{\tau(\beta)}$. In order to show that $\omega_{\beta\cdot\Delta^2}$ induces an holomorphic symplectic structure, as stated in Theorem 1.1.(iii), it suffices to show non-degeneracy, which we now address.

3.2. **Non-degeneracy of $\omega_{\beta\cdot\Delta^2}$.** Let us consider the Mellit chart $\mathfrak{M}$, as constructed in Theorem 2.28, and $\omega_{\mathfrak{M}}$ its corresponding 2-form.[12] In this subsection, we will first show that $\omega_{\mathfrak{M}/T_\pi}$ is non-degenerate, and thus (holomorphic) symplectic. Then we prove, in Theorem 3.5, that ω induces the holomorphic symplectic structure according to Theorem 1.1.(iii).

Following [68, Section 6], we can construct a topological avatar for this torus, as follows. Consider a labeled marked surface $(\mathcal{S}, A, B)$, i.e. an oriented surface $\mathcal{S}$ with boundary $\partial\mathcal{S}$ and two sets $A := \{1, 2, \ldots, n\}, B := \{1', 2', \ldots, n'\} \subseteq \partial\mathcal{S}$ such that:

- Each connected component of $\mathcal{S}$ has a boundary component.
- Each boundary component intersects both A and B.
- The elements of A and B in each boundary component alternate.

Let us denote the two Abelian groups $\Lambda := H_1(\mathcal{S}, A)$ and $\Lambda' := H_1(\mathcal{S}, B)$. Since A and B are alternating, there is a perfect pairing $\cdot : \Lambda \otimes \Lambda' \longrightarrow \mathbb{Z}$. There is also a map $\mathrm{rot} : \Lambda \longrightarrow \Lambda'$, that is induced from the map that, up to homotopy, rotates the boundary components clockwise. This induces a bilinear form $\widetilde{\omega}$ on the first homology Λ, given by $\widetilde{\omega}(\gamma, \gamma') = \gamma \cdot \mathrm{rot}(\gamma')$, and we also consider its anti-symmetrization ω. In order to prove the symplecticity of $\omega_{\beta\cdot\Delta^2}$ stated in Theorem 1.1, we use the following result:

Lemma 3.4 ([68])**.** *There exists a marked surface $(\mathcal{S}, A, B)$ such that Λ is identified with the co-character lattice of $\mathfrak{M}$, and the induced form on the co-character lattice of $\mathfrak{M}$ is identified with ω.*

Note that the surface $\mathcal{S}$ is homeomorphic to the spectral curve constructed in [19], see Section 5.6 for details. Now, we need two more properties of $(\mathcal{S}, A, B)$, which follow from the construction in [68, Section 6.5]:

- The connected components of $\partial\mathcal{S}$ correspond to the cycles of $\pi = \pi(\beta)$, i.e. to the components of the closure of the braid β.
- Let C be the connected component of $\partial\mathcal{S}$ corresponding to the cycle $(a_1 \ldots a_k)$. Then, the elements of $A = \{1, \ldots, n\}$ appearing in C are precisely $a_1, \ldots, a_k$, and they appear in the same order as in the cycle.

Let us denote by $C_1, \ldots, C_k$ the connected components of $\partial\mathcal{S}$. The cycle associated to C_j will be denoted by $(a_{j,1} \ldots a_{j,\ell_j})$. We will now decompose $\Lambda = H_1(\mathcal{S}, A)$, as follows. First, we have the exact sequence in relative homology

$$0 \to H_1(\mathcal{S}) \to H_1(\mathcal{S}, A) \xrightarrow{\partial} H_0(A) = \mathbb{Z}^A \to H_0(\mathcal{S}) \to 0,$$

where the image of ∂ is spanned by elements of the form $a - b$, where $a, b \in A$ belong to the same connected component of $\mathcal{S}$. For each such a, b, we choose a path from a to b in $\mathcal{S}$, and we let K be

[11] The term Darboux-like is used here as the form might not be symplectic at this stage.

[12] Note that the torus $(\mathfrak{M}, \omega_{\mathfrak{M}})$ has already appeared in the work of Mellit [68, Section 6].

the span of the classes of these paths in homology. This gives a splitting

$$H_1(\mathcal{S}, A) = H_1(\mathcal{S}) \oplus K$$

We construct a basis of K as follows. For simplicity, we will assume that $\mathcal{S}$ is connected, the general case follows similarly. For each connected component C_j of $\partial\mathcal{S}$, we take the path from $a_{j,i}$ to $a_{j,i+1}$ following C_j, $j \in [1, \ell_j - 1]$. We also take a path γ_j from a_{j,ℓ_j} to $a_{j+1,1}$, $j \in [1, k-1]$. Then we obtain the basis of K, see Figure 3:

$$K = \mathbb{Z}\{a_{j,i}a_{j,i+1}, \gamma_{j'} \mid j \in [1, k], i \in [1, \ell_j - 1], j' \in [1, k-1]\}.$$

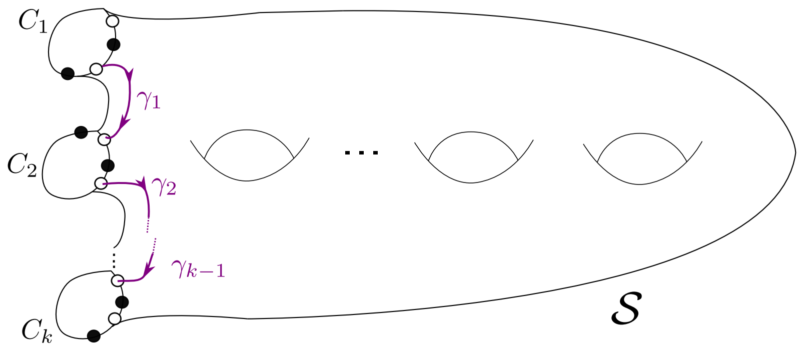


Figure 3. The surface $\mathcal{S}$, with the marked points in the boundary. Points of A are colored white, and points of B are colored black. For the sake of readability we do not label the paths along the boundary for two consecutive points of A.

We can further split $H_1(\mathcal{S})$ as follows. We let $\bar{\mathcal{S}}$ be the surface obtained from $\mathcal{S}$ by attaching disks along the boundary components. We have an exact sequence

$$0 \to H_2(\bar{\mathcal{S}}) \to H_1(\partial\mathcal{S}) \to H_1(\mathcal{S}) \to H_1(\bar{\mathcal{S}}) \to 0$$

so that $H_1(\mathcal{S}) = H_1(\bar{\mathcal{S}}) \oplus (H_1(\partial\mathcal{S})/H_2(\bar{\mathcal{S}}))$. Note that a spanning set for $H_1(\partial\mathcal{S})/H_2(\bar{\mathcal{S}})$ is given by $C_i - C_j$, where C_i and C_j are boundary components of the same connected component of $\mathcal{S}$. Since we are assuming $\mathcal{S}$ is connected, a basis is given by $C_i - C_{i+1}, i \in [1, k-1]$. Moreover, since the elements in $H_1(\mathcal{S})$ are rot-invariant, the form on $H_1(\mathcal{S})$ is given by the intersection form. This implies that $\omega|_{H_1(\bar{\mathcal{S}})}$ is the intersection form on $\bar{\mathcal{S}}$, and therefore is non-degenerate. In addition, $\omega(H_1(\bar{\mathcal{S}}), H_1(\partial\mathcal{S})/H_2(\bar{\mathcal{S}})) = 0$ and $\omega(H_1(\bar{\mathcal{S}}), K) = 0$. Thus, using the decomposition

$$H_1(\mathcal{S}, A) = H_1(\bar{\mathcal{S}}) \oplus (H_1(\partial\mathcal{S})/H_2(\bar{\mathcal{S}})) \oplus K,$$

the form ω has the following form

$$\omega = \begin{pmatrix} \omega|_{H_1(\bar{\mathcal{S}})} & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix}.$$

We will not find the remaining terms * for ω, we will only do so after passing to the quotient by the action of a torus, as this is all that suffices. We have a map $\psi : \mathbb{Z}^A \to H_1(\mathcal{S}, A)$ that to each point $a \in A$ associates the path that follows the boundary component containing a from a to $\mathrm{rot}^2(a)$. In other words, it sends $a_{j,i}$ to a path $a_{j,i}a_{j,i+1}$, where $a_{j,\ell_j+1} = a_{j,1}$.

Now let $T_\pi \subseteq (\mathbb{C}^*)^n/\mathbb{C}^*$ be the torus given by the equations $t_{a_{1,\ell_1}} = t_{a_{2,\ell_2}} = \cdots = t_{a_{k,\ell_k}}$. We recall, see Remark 2.12 and Corollary 2.17, that T_π acts freely on the Mellit chart $\mathfrak{M}$. The torus T_π is the fixed torus for the action of the element $\sigma = (a_{1,\ell_1}a_{2,\ell_2}\cdots a_{k,\ell_k})$. According to [68], to find the co-character lattice of $\mathfrak{M}/T_\pi$ we need to mod out by the image of ψ on σ-invariant elements of A. Thus, the co-character lattice of $\mathfrak{M}/T_\pi$ can be identified with

$$H_1(\bar{\mathcal{S}}) \oplus (H_1(\partial\mathcal{S})/H_2(\bar{\mathcal{S}})) \oplus \mathbb{Z}\{\gamma_1, \dots, \gamma_{k-1}\}.$$

In addition, note that we can identify $C_i = a_{i,\ell_i}a_{i,1}$. Thus, $C_i - C_{i+1} = a_{i,\ell_i}a_{i,1} - a_{i+1,\ell_{i+1}}a_{i+1,1}$. Note also that $\omega(\gamma_i, \gamma_j) = 0$ as $\gamma_i \cdot \mathrm{rot}(\gamma_j) = 0$ for every $i \neq j$. Moreover, $\gamma_i \cdot \mathrm{rot}(a_{i,\ell_i}a_{i,1} - a_{i+1,\ell_{i+1}}a_{i+1,1}) = 0$ while $(a_{i,\ell_i}a_{i,1} - a_{i+1,\ell_{i+1}}a_{i+1,1}) \cdot \gamma_i = 2$. Thus, $\omega(\gamma_i, a_{i,\ell_i}a_{i,1} - a_{i+1,\ell_{i+1}}a_{i+1,1}) = 2$. Similarly, we can see that $\omega(\gamma_i, a_{i-1,\ell_{i-1}}a_{i-1,1} - a_{i,\ell_i}a_{i,1}) = 1$ and $\omega(\gamma_i, a_{i+1,\ell_{i+1}}a_{i+1,1} - a_{i+2,\ell_{i+2}}a_{i+2,1}) = -1$. It follows that the form $\omega_{\mathfrak{M}/T_\pi}$ is given by the following matrix

$$\omega_{\mathfrak{M}/T_\pi} = \begin{pmatrix} \omega|_{H_1(\bar{S})} & 0 & 0 \\ 0 & 0 & -P \\ 0 & P & 0 \end{pmatrix},$$

where P is the $(k-1)\times(k-1)$-matrix

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

This implies that the 2-form $\omega_{\mathfrak{M}/T_\pi}$ is non-degenerate and thus the chart $\mathfrak{M}/T_\pi$ is (holomorphic) symplectic. Let us now use the above discussion, and this result for the Mellit chart, to conclude Theorem 1.1.(iii).

Theorem 3.5. *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid (word). Then, the 2-form on $\omega_{\beta\cdot\Delta^2}$ induces a 2-form on the augmentation variety $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ that has maximal rank at every point. Thus, the augmentation variety of any positive braid is holomorphic symplectic.*

Proof. The augmentation variety $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ can be identified with $X_0(\beta\cdot\Delta; w_0)/T_\pi$, up to the choice of marked points $\mathfrak{t}_c$. The coefficients of the form $\omega_{\beta\cdot\Delta^2}$ are regular functions on $X_0(\beta\cdot\Delta^2)$, and it is readily verified that the form is T-equivariant. Thus, we have an induced closed 2-form ω_β on the augmentation variety, and it is non-degenerate if and only if its determinant does not vanish anywhere. Let us first prove that it is non-degenerate on all toric charts. Thanks to the discussion above on the Mellit chart, the form ω_β is non-degenerate on the (quotient) toric chart $\mathfrak{M}/T_\pi$; recall that $\mathfrak{M}$ is the toric chart obtained from opening the crossings in the order given by (the proof of) Theorem 2.28. By Lemma 3.3, the 2-form has constant coefficients in canonical coordinates in any other chart $\mathfrak{M}'/T_\pi$ and, by the above, it is non-degenerate on the intersection with $\mathfrak{M}/T_\pi$. Thus, the 2-form is non-degenerate on the entire (other) chart $\mathfrak{M}'/T_\pi$. Finally, by Theorem 2.18, the toric charts cover $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ up to codimension 2. Hence, the determinant of ω_β is non-zero outside of codimension 2 locus and hence it is non-zero everywhere. □

This concludes the proof of Theorem 1.1 and establishes that the augmentation variety associated to a positive braid is holomorphic symplectic. This subsection also concludes the first part of the article, and we now move forward to discuss correspondences between braid varieties and the diagrammatic calculus we develop for their study.

4. The Combinatorics of Algebraic Weaves

This section introduces algebraic weaves, a diagrammatic calculus to study the braid varieties $X_0(\beta)$. The present section focuses on the combinatorial aspects of these diagrams; in particular, this formalizes the weave category $\mathfrak{W}_n$ discussed in Section 1. We use these algebraic weaves in Section 5, where we prove that a weave between two positive braid β_1 and β_2 yields a correspondence between the braid varieties $X_0(\beta_1)$ and $X_0(\beta_2)$ (as stated in Theorem 1.4). We refer the reader to [19] for the contact and symplectic geometry motivation behind algebraic weaves.

4.1. Algebraic weaves.

Algebraic weaves are planar diagrams introduced in the work of the first author and E. Zaslow [19]. In appearance, these are similar to the planar diagrams appearing in Soergel calculus [30, 31]; there are nevertheless key distinctions, and we refer to our diagrams as *weaves*, following the symplectic constructions in [19].

Definition 4.1. Let β_1, β_2 be two positive n-braid words. A weave of degree n is the image of a continuous map

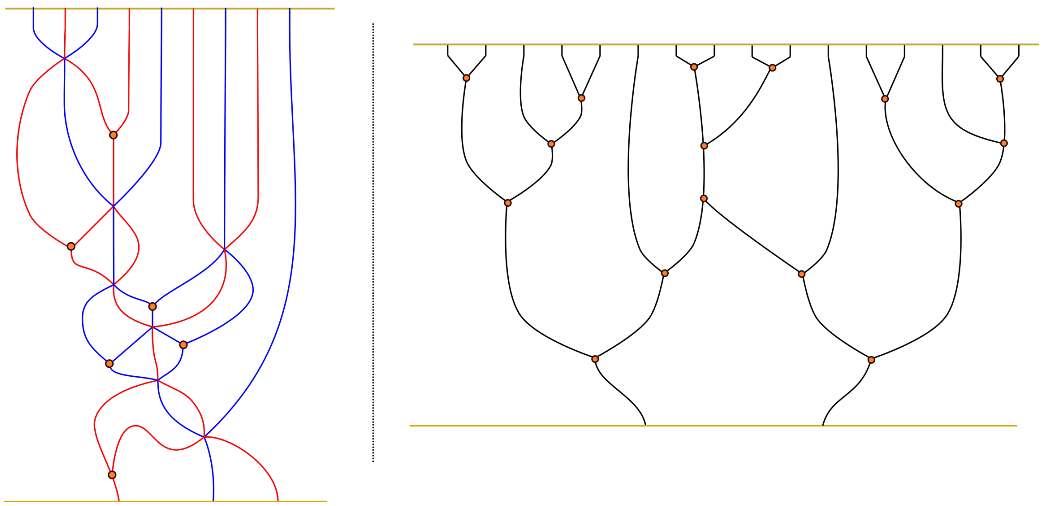


FIGURE 4. A 3-weave from $\beta_2 = (\sigma_1\sigma_2)^4\sigma_1 \in \mathrm{Br}_3^+$ to $\beta_1 = \sigma_2\sigma_1\sigma_2 \in Br_3^+$ (Left). The blue color indicates a transposition label $s_1 \in S_3$ and the red color indicates the transposition label $s_2 \in S_3$. A 2-weave from $\beta_2 = \sigma_1^{16} \in \mathrm{Br}_2^+$ to $\beta_1 = \sigma_1^2 \in \mathrm{Br}_2^+$, all black edges are labeled with the unique transposition $s_1 \in S^2$.

$$\mathfrak{w} : \bigcup_{i=1}^{n-1} G_i \longrightarrow [1,2] \times \mathbb{R},$$

where each G_i, $i \in [1, n-1]$ is a trivalent graph and the following conditions are satisfied:

(i) The restriction $\mathfrak{w}|_{G_i} : G_i \longrightarrow [1,2] \times \mathbb{R}$ is a topological embedding for all $i \in [1, n-1]$, which is a smooth embedding away from the trivalent vertices of the graph G_i.

(ii) The images $\mathfrak{w}(G_i)$ and $\mathfrak{w}(G_{i+1})$ are only allowed to intersect at trivalent vertices, $i \in [1, n-2]$, and the planar edges around this intersection point must alternatingly belong to G_i and G_{i+1}. In addition, intersections between $\mathfrak{w}(G_i)$ and $\mathfrak{w}(G_j)$, $|i-j| \geq 2$ are transverse.

(iii) In a neighborhood of $\{j\} \times \mathbb{R} \subseteq [1,2] \times \mathbb{R}$, $j = 1, 2$, the image $im(\mathfrak{w})$ is given by $l(\beta_j)$ vertical lines, such that the kth line belongs to $G_{\sigma^{(j)}_{i_k}}$, where $\sigma^{(j)}_{i_k}$ is the kth crossing of β_j. $\square$

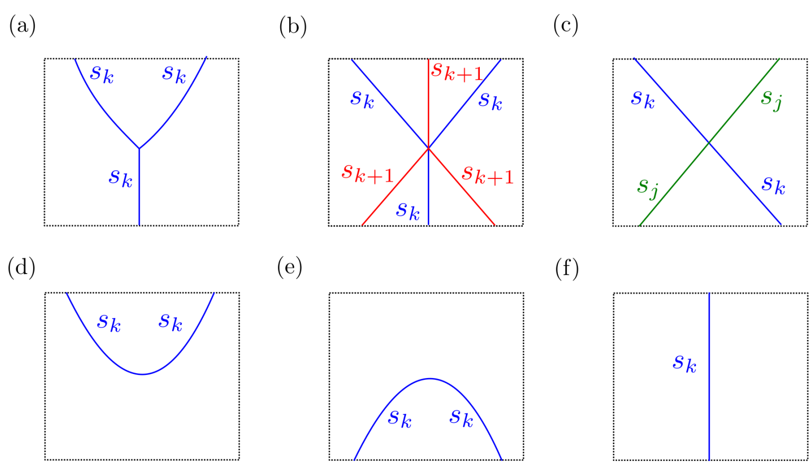


FIGURE 5. The six local models for an algebraic n-weave, where $j, k \in [1, n-1]$, except in (b) where $k \in [1, n-2]$, and $|j-k| \geq 2$. The inverse of the local model in (b), with $s_{k+1}s_ks_{k+1}$ on top and $s_ks_{k+1}s_k$ on the bottom, is also allowed.

Definition 4.1 is precise, but admittedly obtuse; see Figure 4 for two explicit examples with $n = 2, 3$. Intuitively, we start with β_2 being represented by a collection of points in $\{2\}\times\mathbb{R}$, e.g. $(2, k) \in \{2\}\times\mathbb{R}$, $k \in [1, \ell(\beta_2)]$ where we label the kth point with the transposition $s_k \in S_{\ell(\beta_2)}$ associated to the kth crossing of β_2. Then we start drawing lines going down, towards $\{1\} \times \mathbb{R}$, from each of the points $(2, k)$, which are the crossings of β_2. Each of these lines, which we will call *edges*, are labeled by the same transposition s_k as their starting (highest) point. As we keep drawing the edges downwards, one of the following six situations, depicted in Figure 5, might occur:

(a) Two consecutive edges labeled with the *same* transposition s_k come together, and continue moving down as one unique edge, also labeled with s_k, $k \in [1, n-1]$. This is referred to as a *trivalent vertex*, and correspond to the model around (the image of) a trivalent vertex of the graph G_k in Definition 4.1. Algebraically, we represent this local model by $s_k s_k \to s_k$.

(b) Three consecutive edges labeled by s_k, s_{k+1}, s_k come together, and continue moving down as three edges but now labeled s_{k+1}, s_k, s_{k+1}. This is referred to as a *hexavalent vertex*, and correspond to the model around an intersection point of the (images of the) graphs $\mathfrak{w}(G_k)\cap\mathfrak{w}(G_{k+1})$ in Definition 4.1. Algebraically, we represent this local model by $s_k s_{k+1} s_k \to s_{k+1} s_k s_{k+1}$. In addition, we also allow the same move, but reversed: $s_{k+1} s_k s_{k+1} \to s_k s_{k+1} s_k$, with $s_{k+1} s_k s_{k+1}$ on top and $s_k s_{k+1} s_k$ at the bottom.

(c) Two consecutive edges labeled with two different transpositions s_k, s_j, with $|j - k| \geq 2$, come together, and continue moving down as two edges, now labeled by s_j, s_k. This is referred to as a *4-valent vertex*, and correspond to the model around a (transverse) intersection point of $\mathfrak{w}(G_k)$ and $\mathfrak{w}(G_j)$ in Definition 4.1. Algebraically, we represent this local model by $s_k s_j \to s_j s_k$.

(d) Two consecutive edges labeled with the *same* transposition s_k come together, merge and there is no edge continuing down. This is referred to as a *cup*, and we represent this local model by $s_k s_k \to 1$.

(e) The inverse of the move in (d), where two consecutive edges are created as moving downwards from the empty set. This is referred to as a *cap*, and we represent this local model by $1 \to s_k s_k$.

(f) There is an edge labeled by s_k and it continues moving down as the same edge labeled by s_k, i.e. nothing occurs. This local model is represented algebraically by $s_k \to s_k$.

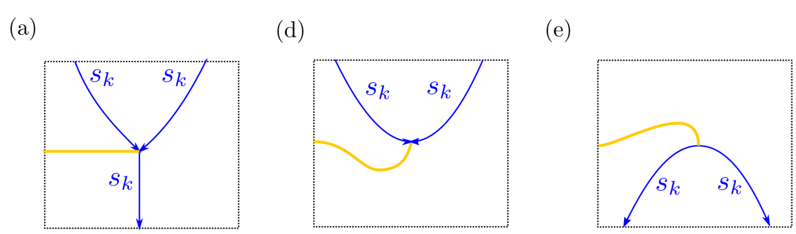


FIGURE 6. Local models for weaves with yellow segments.

In the six models (a)-(f) above, the algebraic representation is merely the result of taking horizontal cross-sections along the weave (or local model) and reading the permutations s_k (or braid group generators σ_k), left to right, that label the edges intersecting the cross-section. For instance, four-valent and hexavalent vertices represent (the Coxeter projection of the) braid relations. In the course of the manuscript, we will also use *yellow segments* in order to keep track of certain variables, corresponding to the z_i-variables in the braid variety. In addition, we also consider particular types of weaves. This is the content of the following:

Definition 4.2. An algebraic weave is a weave such that:

- The edges have been oriented downwards, with the models according to Figure 6 for cups and caps. By convention, diagrams are oriented from top to bottom, from $\{2\}\times\mathbb{R}$ down to $\{1\}\times\mathbb{R}$.

- Trivalent vertices, cups and caps are decorated with yellow segments to the left.[13] See Figure 6 for a depiction of these yellow segments. Near another vertex, we allow a yellow line to intersect either all incoming or all outgoing edges.

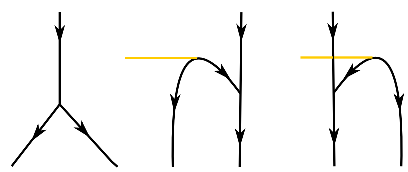
The following two are special types of weaves that we will use:

(ii) A simplifying weave is an algebraic weave with no caps; thus the only allowed local models are (a),(b),(c),(d) and (f), not (e).

(ii) A Demazure weave is an algebraic weave with no cups nor caps; thus the only allowed local models are (a),(b),(c) and (f), not (d),(e).

It is readily seen that an algebraic weave is simplifying if and only if the length of a braid word is not increasing as we scan down the weave with horizontal cross-sections. The reasons behind the choice of the name "Demazure weaves" will be explained in Section 4.3. □

Note that upside-down trivalent vertices, i.e. the horizontal flip of model (a), given by $s_k \to s_k s_k$, can be constructed using the above trivalent vertices and caps. In fact, an upside-down trivalent vertex can be created in at least the following ways:



In the Subsection 4.2, these different ways of drawing upside-down vertices are declared to be equivalent. Finally, an (algebraic) n-weave Σ will almost always be considered through its horizontal cross-sections, and thus we will typically refer to it as being a sequence of (consecutively distinct) positive braid words

$$\beta_j(\Sigma) = s_{i_1}^{(j)} s_{i_2}^{(j)} \cdots s_{i_{\ell(\beta_j)}}^{(j)} \in \mathrm{Br}_n^+, \quad j \in [0, \ell(\Sigma)],$$

where the only changes are the ones specified in the local models (a)-(f) above. In this case, we use the notation

$$\Sigma = \beta_0(\Sigma) \to \beta_1(\Sigma) \to \cdots \to \beta_{\ell(\Sigma)}(\Sigma).$$

The initial braid word $\beta_0(\Sigma)$ being read at the horizontal cross-section $\{2\}\times\mathbb{R}$, and the last braid word $\beta_{\ell(\Sigma)}(\Sigma)$ is read at the horizontal cross-section $\{1\}\times\mathbb{R}$. The number $\ell(\Sigma)\in\mathbb{N}$ will be referred to as the length of the weave Σ, the initial braid word $\beta_0(\Sigma)$ as the convex end of the weave, and the last braid word $\beta_{\ell(\Sigma)}(\Sigma)$ as the concave end of the weave. The convexity and concavity notation aims at agreeing with the contact topological framework developed in [19].

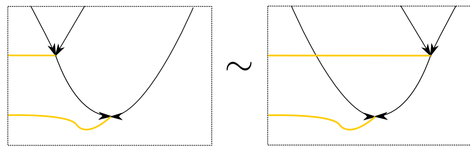
4.2. **Equivalence of weaves.** In this section, we impose certain equivalences between two algebraic weaves Σ_1, Σ_2 whose convex and concave ends coincide, i.e. $\beta_0(\Sigma_1) = \beta_0(\Sigma_2)$ and $\beta_{\ell(\Sigma_1)}(\Sigma_1) = \beta_{\ell(\Sigma_2)}(\Sigma_2)$. These equivalences $\Sigma_1 \simeq \Sigma_2$ are the result of imposing a Hamiltonian isotopy equivalence to the (exact) Lagrangian surfaces obtained by projecting to $(T^*([1,2]\times\mathbb{R}^2), d\lambda_{\mathrm{st}})$ the Legendrian lifts of these weaves Σ_1, Σ_2 in the standard contact 5-dimensional Darboux chart $(J^1([1,2]\times\mathbb{R}^2), \xi_{\mathrm{st}})$. The geometric proofs are explained in detail in [19, Section 4]. In the present article, we directly write the diagrammatic equivalences, which can be understood as moves between braid isotopies (also known as movies moves).

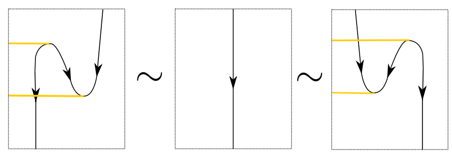
Remark 4.3. These equivalences are chosen as dictated by contact topology but can be now taken to be diagrammatic. The remarkable algebraic properties of these equivalences that we list below is that the braid matrices that we will associate to the weaves in Section 5 will in fact preserve these equivalences. In particular, an algebraic reason *not* to declare the mutated weaves equivalent is that they will not preserve these braid matrix relations. □

[13]These yellow segments are not considered part of the graph, and in particular they are not edges, i.e. a trivalent vertex is still considered trivalent even if there is a yellow segment from it.

Each of the following items represents an equivalence between weaves. The majority of equivalences we describe compare two local models, and an equivalence between two different weaves Σ_1, Σ_2 will be obtained by applying several of the local equivalences listed here.

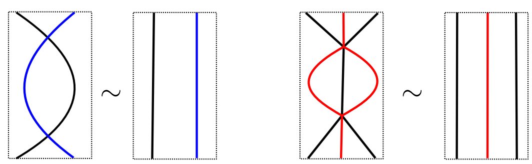
4.2.1. *Planar isotopies.* A weave is, in particular, a planar diagram: we declare planar isotopic diagrams to define equivalent weaves. As a first consequence, the two ways to declare an upside down trivalent vertex are the same, and we have the following additional relations:





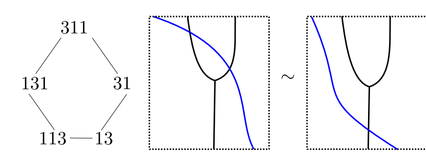
These planar isotopies do include canceling pairs of caps and cup, as depicted above. For instance, the weave $\Sigma_1 = s_k \to s_k s_k s_k \to s_k \cdot 1 = s_k$, where first a cap creates $1 \to (s_k s_k)$ to the left of the initial s_k, and then a cup erases the rightmost to s_k via $(s_k s_k) \to 1$, is equivalent to the constant weave $\Sigma_2 = s_k$.

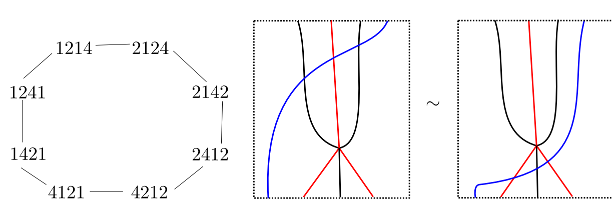
4.2.2. *Canceling pairs of 4- and 6-valent vertices.* The following weaves are declared to be equivalent:



From the algebraic perspective, i.e. studying the braids in the horizontal cross-sections, this the diagrammatic incarnation of the fact that the two moves $s_k s_{k+1} s_k \to s_{k+1} s_k s_{k+1}$ and $s_{k+1} s_k s_{k+1} \to s_k s_{k+1} s_k$, and the two moves $s_i s_j \to s_j s_i$ and $s_j s_i \to s_i s_j$, $|i-j| \geq 2$, are inverse to each other. That is, performing a Reidemeister III move and then its inverse is considered to be (equivalent to) the trivial weave. Similarly, performing a commutation move in the braid group, and then the same move in reverse, is also considered to be (equivalent to) the trivial weave. In the notation above, we are declaring the weave $\Sigma_1 = s_{k+1} s_k s_{k+1} \to s_k s_{k+1} s_k \to s_{k+1} s_k s_{k+1}$ to be equivalent to the constant weave $\Sigma_2 = s_{k+1} s_k s_{k+1}$, and the weave $\Sigma_1 = s_i s_j \to s_j s_i \to s_i s_j$ to be equivalent to the constant weave $\Sigma_2 = s_i s_j$.

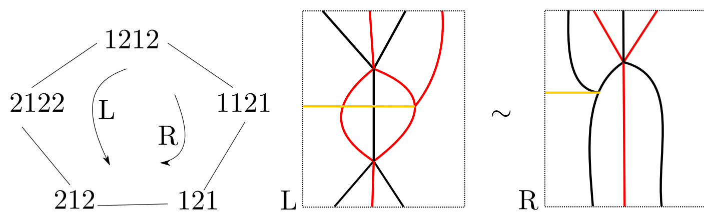
4.2.3. *Commutation with distant colors.* We declare that an edge of the weave labeled with a color (i.e. a transposition) which is distant to the rest of the colors at a given vertex can be moved past this vertex. That is, we declare that the following weaves are equivalent:



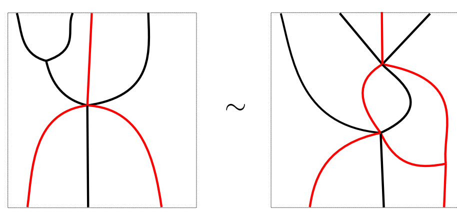


Similarly, we declare that three lines with pairwise distant colors can be rearranged according to the weave equivalence depicted above. As illustrated in the equivalences above, the particular sequences of braid moves that we are declaring to be equivalent are readily read from taking horizontal cross-sections in the above diagrams; we will thus not indicate them any longer.

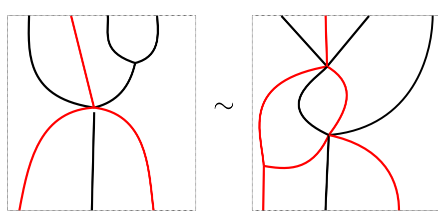
4.2.4. *1212- and 2121-relations.* We require that the following two ways of getting from $\sigma_1\sigma_2\sigma_1\sigma_2$, denoted 1212 for simplicity, to $\sigma_1\sigma_2\sigma_1$, i.e. 121, are equivalent:



Note that we interpret the same pentagon, on the left of the above figure, in several different ways. Namely, this requirement also means that the two ways of getting from 1121 to 212 are equivalent, and that the two ways of getting from 2212 to 121 are equivalent, and so forth. The equivalence of the two ways of getting from 1121 to 212 corresponds to the equivalence of the following two simplifying weaves:

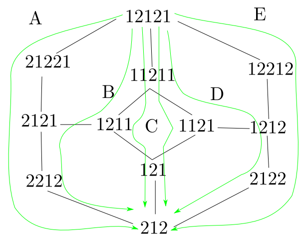


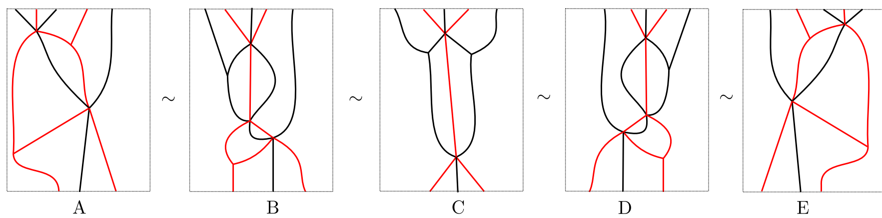
We also require that the two ways of getting from 1211 to 212 are equivalent, which is the same as requiring that the two ways of getting from 2121 to 212 are equivalent and so on. The weaves are obtained from the ones above by the symmetry along the vertical line:



There are also similar relations where adjacent colors are interchanged (e.g. black and red), and for any pair of adjacent colors, which we do not draw them here.

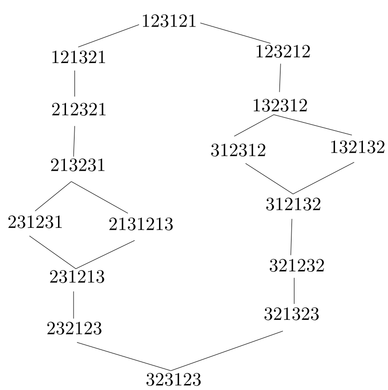
4.2.5. *Cycles for 12121.* As an example for the previous relation, we observe that there are many paths in the Demazure graph from 12121 to 212, related by consecutive application of the 1212-relation:

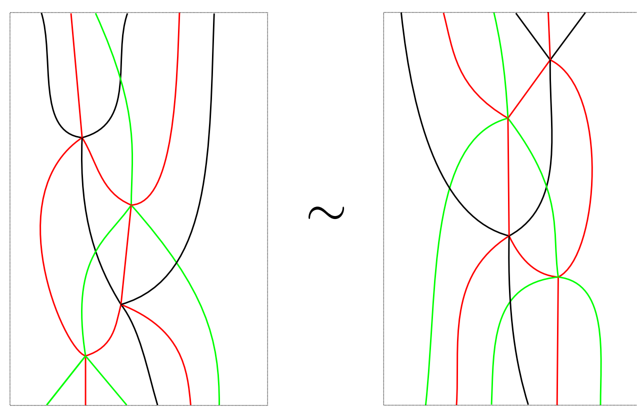




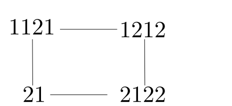
$$(A)\ 12121 \to 21221 \to 2121 \to 2212 \to 212$$
$$(B)\ 12121 \to 11211 \to 1211 \to 2121 \to 2112 \to 212$$
$$(C)\ 12121 \to 11211 \to 1211 \to 121 \to 212 \sim 12121 \to 11211 \to 1121 \to 121 \to 212$$
$$(D)\ 12121 \to 11211 \to 1121 \to 1212 \to 2122 \to 212$$
$$(E)\ 12121 \to 12212 \to 1212 \to 2122 \to 212$$

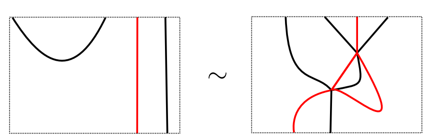
4.2.6. *Zamolodchikov relation.* Diagrammatically, the Zamolodchikov relation is the equivalence of the following diagrams, relating various braid words for the longest element $w_0 \in S_4$:

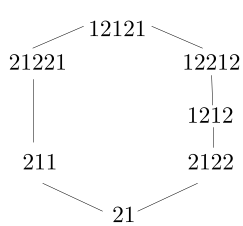


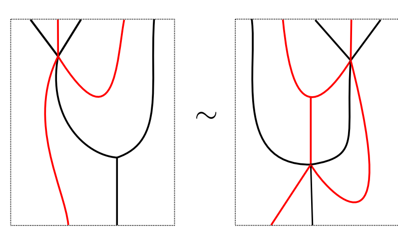


4.2.7. *More moves with cups.* By rotating the above relations, we obtain many interesting relations with cups and caps. We include just a few here:

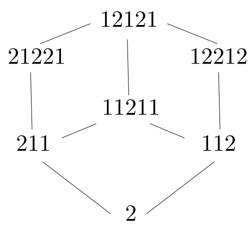


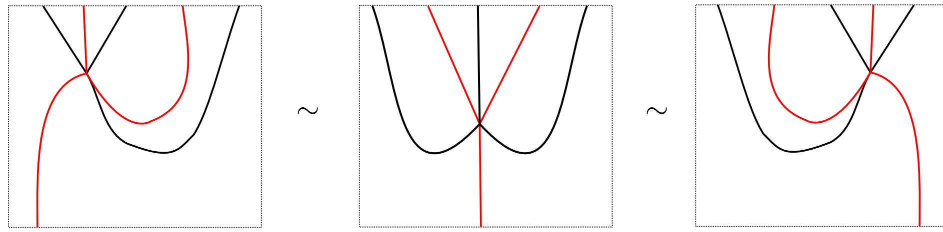




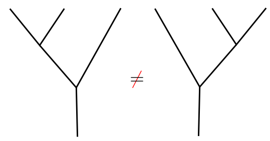


Note that these two pictures, as planar graphs, are similar to the ones that we already considered (cancellation $121 \to 212 \to 121$ and 1212-move), but in this case they are drawn differently. The following relations corresponds to different ways to a look at a single 6-valent vertex:





4.2.8. *Mutations.* In contrast with Soergel calculus, we do not declare two ways of getting from $s_is_is_i$ to s_i via s_is_i to be equivalent. They are related by the following special type of move, which we call a weave *mutation*:



This concludes the list of diagrammatic equivalences (4.2.1)-(4.2.6), and the mutation non-equivalence (4.2.8).

4.3. **Demazure product and Demazure weaves.** The NilHecke algebra NH_n is defined by generators $e_1, \dots, e_{n-1}$ and relations $e_i^2 = e_i$ and

$$e_ie_{i+1}e_i = e_{i+1}e_ie_{i+1}, \quad e_ie_j = e_je_i, \ \ |i-j| \geq 2.$$

We have a natural homomorphism from the braid group to NH_n which we will denote by δ following [60]. We will denote by e_w the image of the positive reduced braid lift of $w \in S_n$ under δ. It is clear that for any braid β there exists w such that $\delta(\beta) = e_w$. By abuse of notation, we will write $\delta(\beta) = w$ and call w the *Demazure product* of β. This terminology is used by Knutson and Miller [60], but the notion goes back at least to Demazure [26]. Given two permutations $u, v \in S_n$, we define $u \star v = \delta(uv)$. By construction, we have

$$(u \star v) \star w = \delta(uvw) = u \star (v \star w).$$

We will call the star product of the Coxeter generators corresponding to the letters in a positive braid word the *Demazure product* of the word. For a braid word $\sigma_{i_1}\sigma_{i_1}\dots\sigma_{i_{l(\beta)}}$ for a positive braid β, we have

$$e_{i_1} \star e_{i_2} \star \dots \star e_{i_{l(\beta)}} = \delta(\beta).$$

In other words, the Demazure product of a braid word for β equals $\delta(\beta)$. In particular, it does not depend on the choice of a word for β.

Example 4.4. *It is readily verified that $w \star s_i = ws_i$ if $\ell(ws_i) = \ell(w) + 1$ and $w \star s_i = w$ if $\ell(ws_i) = \ell(w) - 1$. The Demazure power is simply $s_i \star s_i \star \dots \star s_i = s_i$, for any number of multiples, and we also have the equality $w \star w_0 = w_0$.* □

The NilHecke algebra NH_n is the monoid algebra of the monoid defined by the same generators and relations, whose multiplication is given by the Demazure product. For this reason, the latter is also known as the *0-Hecke product*; the monoid is called *Coxeter monoid* [89], $0-$*Hecke monoid* [34, 48], or *Richardson-Springer monoid* by different authors. Richardson and Springer studied its action on the set of orbits of the flag variety under the action of the fixed point subgroup of an involution on the algebraic group [80, 81]. The elements of this monoid are precisely $\{e_w | w \in S_n\}$, i.e. we have a bijection between the set S_n and the underlying set of the monoid (this was proved by Norton [73]). As the examples above show, the multiplication in the monoid is quite different from the one in the permutation group. More generally, given a positive braid β, $\delta(\beta)$ does not coincide with the image of β under the canonical surjection onto S_n. A first relation to the weaves introduced above is given in the following simple:

Lemma 4.5. *Let Σ be a Demazure weave. Then the Demazure product of the associated braid words $\beta_j(\Sigma)$, $j \in [0, l(\Sigma)]$, remains unchanged, i.e. $\delta(\beta_0(\Sigma)) = \delta(\beta_j(\Sigma))$.*

Proof. By the arguments above, the Demazure product is invariant under braid moves in a word, so 6- and 4-valent vertices do not change it. Example 4.4 shows that 3-valent vertices also preserve the Demazure product. □

Lemma 4.5 shows that Demazure weaves provide a transparent diagrammatic interpretation of the Demazure product and of the 0−Hecke monoid. This motivated our nomenclature.

4.4. **Classification of algebraic weaves.** We call two algebraic weaves *equivalent* if they are related by a sequence of elementary equivalence moves from the previous section (with no mutations), and *mutation equivalent* if they are related by a sequence of equivalence moves and mutations.

Theorem 4.6. (*a*) *Let Σ_1, Σ_2 be two algebraic weaves with the same source and target braids. If Σ_1, Σ_2 only have 6- and 4- valent vertices, then Σ_1, Σ_2 are equivalent.*

(*b*) *Let Σ_1, Σ_2 be two* Demazure *algebraic weaves with the same source and target. If the target is reduced, then Σ_1, Σ_2 are mutation equivalent.*

Proof. The theorem follows from the main result of [29], which we briefly recall. For part (a), consider the graph where vertices correspond to braid words and edges to braid moves (that is, 6- or 4-valent vertices). Then the cycles in this graph are generated by commutation with distant colors and Zamolodchikov relations, hence any two paths in this graph are equivalent.

For (b), consider the Hecke-type algebra with generators T_i and relations

$$T_i^2 = \alpha T_i + \beta,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} + \text{lower order terms},\ T_iT_j = T_jT_i, (|i-j| > 1).$$

Using these relations, it is easy to see that every product of T_i can be written as a linear combination of reduced expressions, possibly in a non-unique way. This non-uniqueness appears from *ambiguities*: applying the relations in different order could yield a different results.

B. Elias proved in [29, Proposition 5.5] that (modulo commutation with distant colors) there are exactly 5 types of potential abmiguities that one needs to consider: $iii, ii(i+1)i, i(i+1)ii, i(i+1)i(i+1)i, i(i+1)(i+2)i(i+1)i$, which are nothing but the trivial move, the 5-cycles corresponding to 1121 and 1211 from 4.2.4, the cycle from 4.2.5 for the word 12121, and the Zamolodchikov relation. Note that the ambiguity iii corresponds to different ways of getting from iii to i. There are two such ways without cups, and they are related by the first mutation from Section 4.2.8. □

Remark 4.7. The assumption in (b) that the target is reduced is important. Indeed, the two trivalent vertices $(ss)s \to ss$ and $s(ss) \to ss$ are not mutation equivalent.

Remark 4.8. Note that by Theorem 4.6(a), any two simplifying weaves relating two braid words for the same braid are equivalent. Thus, we will not specify such a weave.

Let us continue studying conditions for equivalences, it is indeed useful to have different criteria when verifying that two weaves are equivalent (or mutation equivalent). Suppose that a braid word β contains a piece s_ius_j. Following [47], the pair of crossings (s_i, s_j) is said to be a *deletion pair* if $s_iu = us_j$.[14] Let us define a relation $\prec$ on the crossings of the braid β according to $s_i \prec s_j$ if (s_i, s_j) form a deletion pair. The following two lemmas are used in the proof of the criterion Theorem 4.11 below:

Lemma 4.9. (i) *The relation $\prec$ is a partial order on the set of crossings of β.*
(ii) *The set of crossings of β is a disjoint union of linearly ordered sets.*

Proof. Assume that we have a piece of a braid $s_ius_jvs_k$ and (s_i, s_j) and (s_j, s_k) are deletion pairs so that $s_iu = us_j, s_jv = vs_k$. Then

$$s_i \cdot us_jv = us_js_jv = us_jv \cdot s_k,$$

and (s_i, s_k) is a deletion pair. This proves (i). To prove (ii), assume (s_i, s_j) and (s_i, s_k) are deletion pairs, and assume wlog that s_j is to the left of s_k. We must show that (s_j, s_k) is a deletion pair. We have

$$s_iu = us_j,\ s_ius_jv = us_jvs_k$$

[14] Note that unlike [47], we do not require u to be a reduced word.

and $s_ius_jv = s_iuvs_k$, hence $s_jv = vs_k$ and (s_j, s_k) is a deletion pair. The case when (s_i, s_k) and (s_j, s_k) are deletion pairs is analogous. □

We call a deletion pair (s_i, s_j) *close* if $s_i \prec s_j$ is a cover relation, i.e. no crossing in-between s_i and s_j forms a deletion pair with s_i or s_j.

Lemma 4.10. *Suppose that (s_i, s_j) is a close deletion pair, then the following Demazure weaves are equivalent:*

$$s_ius_j \to s_is_iu \to s_iu \to us_j \ \sim \ s_ius_j \to us_js_j \to us_j.$$

Note that the condition that the deletion pair is close is necessary, see Remark 4.7.

Proof. We prove the statement by induction by the length of u. If u is empty, the statement is clear. Otherwise, by definition of deletion pair we get $s_iu = us_j$. If u ends with s_j then we do not have a close pair, contradiction. Otherwise we need to apply some braid relation to us_j which involves s_j. We have the following cases:

1) If $u = vs_k$ and $|k - j| > 1$ then $us_j = vs_ks_j = vs_js_k$ while $s_iu = s_ivs_k$, so $vs_j = s_iv$. By the assumption of induction, two Demazure weaves corresponding to $s_ivs_js_k$ are equivalent, and we get the following diagram:

$$\begin{tikzcd}
s_ivs_ks_j \arrow[r] \arrow[d] & vs_js_ks_j \arrow[r] \arrow[d] & vs_ks_js_j \arrow[d] \\
s_ivs_js_k \arrow[r] \arrow[d] & vs_js_js_k \arrow[dr] & vs_ks_j \\
s_is_ivs_k \arrow[r] & s_ivs_k \arrow[r] & vs_js_k \arrow[u]
\end{tikzcd}$$

The top square is an isotopy, and the pentagon on the right is commutation with distant colors.

2) If $u = vs_js_{j+1}$ then $us_j = vs_js_{j+1}s_j = vs_{j+1}s_js_{j+1}$ while $s_iu = s_ivs_js_{j+1}$, so $s_iv = vs_{j+1}$. By the assumption of induction, two Demazure weaves corresponding to s_ivs_{j+1} are equivalent, and we get the following diagram:

$$\begin{tikzcd}
s_ivs_js_{j+1}s_j \arrow[r] \arrow[d] & vs_{j+1}s_js_{j+1}s_j \arrow[r] \arrow[d] & vs_js_{j+1}s_js_j \arrow[d] \\
s_ivs_{j+1}s_js_{j+1} \arrow[r] \arrow[d] & vs_{j+1}s_{j+1}s_js_{j+1} \arrow[dr] & vs_js_{j+1}s_j \\
s_is_ivs_js_{j+1} \arrow[r] & s_ivs_js_{j+1} \arrow[r] & vs_{j+1}s_js_{j+1} \arrow[u]
\end{tikzcd}$$

The top square is an isotopy, and the pentagon on the right is 5-cycle from Section 4.2.4. The case when $u = vs_js_{j-1}$ is analogous. □

Given a Demazure weave Σ, we have an injection ι_Σ from the set of crossings in the target to the set of crossings in the top: for a 6-valent vertex it is a bijection which exchanges left and right crossings, for a 4-valent vertex it is a bijection exchanging crossings, and for a 3-valent vertex the injection sends the crossing in the target to the right crossing in the source. We refer to the crossings not in the image of ι_Σ as *missing*. The following result is a characterization of the equivalence between Demazure weaves:

Theorem 4.11. *Suppose that $\ell(\beta_1) = \ell(\beta_0) - 1$. Then two Demazure weaves from β_0 to β_1 are equivalent if and only if they have the same missing crossing in β_0.*

Proof. Since $\ell(\beta_1) = \ell(\beta_0) - 1$, any Demazure weave between β_0 and β_1 has one trivalent vertex. Let us prove that equivalent weaves have the same missing crossing. It is easy to see that commutations with distant colors and Zamolodchikov relations induce the same bijections between crossings, so any two weaves with the same source and target and only 6- and 4-valent vertices induce the same bijection. Finally, for the 5-cycle from Section 4.2.4 we observe that in either weave for 1121 the first crossing is missing, while in either weave for 1211 the third crossing is missing.

Conversely, assume that the Demazure weaves $\Sigma_1, \Sigma_2 : \beta_0 \to \beta_1$ have the same missing crossing. It is sufficient to prove that they can be related by a sequence of cycles from Lemma 4.10 and equivalences. Note that a trivalent vertex corresponds to a close deletion pair. We have the following cases, where the deletion pair is underlined:

1) Assume that (s_i, s_j) is a close deletion pair and we apply a 4-valent vertex to s_j:
$$\underline{s_i}u\underline{s_j}s_k = \underline{s_i}us_k\underline{s_j},\ |k-j|>1,$$
then (s_i, s_j) is a close deletion pair in the resulting braid, and Lemma 4.10 applies.
2) Assume that we apply a 6-valent vertex with s_j on the left:
$$\underline{s_i}u\underline{s_j}s_{j+1}s_j = \underline{s_i}us_{j+1}s_j\underline{s_{j+1}}$$
then (s_i, s_{j+1}) is a close deletion pair in the resulting braid, and Lemma 4.10 applies.
3) Assume that we apply a 6-valent vertex with s_j on the right:
$$\underline{s_i}us_js_{j+1}\underline{s_j} = \underline{s_i}u\underline{s_{j+1}}s_js_{j+1}$$
Note that $s_ius_js_{j+1} = us_js_{j+1}s_j = us_{j+1}s_js_{j+1}$ implies $s_iu = us_{j+1}$, and (s_i, s_{j+1}) is again a close deletion pair.
4) Finally, assume that we apply a 6-valent vertex with s_j in the middle, then we no longer get a deletion pair. Instead, $u = vs_{j+1}$ and $s_ivs_{j+1} = vs_{j+1}s_j$. By considering possible braid moves, we can write $v = ws_j$, then
$$s_iws_js_{j+1} = ws_js_{j+1}s_j = ws_{j+1}s_js_{j+1},$$
hence $s_iw = ws_{j+1}$.We get the following diagram:
$$\begin{tikzcd}
\underline{s_i}ws_js_{j+1}\underline{s_j}s_{j+1} \arrow[r] \arrow[d] \arrow[dr] & \underline{s_i}ws_js_j\underline{s_{j+1}}s_j \arrow[r] & w\underline{s_{j+1}}s_js_j\underline{s_{j+1}}s_j \arrow[dl] \\
\underline{s_i}w\underline{s_{j+1}}s_js_{j+1}s_{j+1} \arrow[d] & w\underline{s_{j+1}}s_js_{j+1}\underline{s_j}s_{j+1} \arrow[r] \arrow[dl] & ws_js_{j+1}\underline{s_js_j}s_{j+1} \arrow[d] \\
w\underline{s_{j+1}s_{j+1}}s_js_{j+1}s_{j+1} \arrow[r] & ws_{j+1}s_js_{j+1}s_{j+1} \arrow[r] & ws_js_{j+1}s_js_{j+1}
\end{tikzcd}$$
Here the squares are isotopies and 5-cycle is an equivalence from Section 4.2.4.

By combining all these cases (and the ones obtained by changing $j+1$ to $j-1$, or applying braid moves to s_i), we can find equivalent weaves $\Sigma_1 \sim \Sigma_1'' \circ \Sigma_1'$ and $\Sigma_2 \sim \Sigma_2'' \circ \Sigma_2'$ where

- $\Sigma_1', \Sigma_2' : \beta_0 \to \beta'$ are weaves between equivalent braid words.
- $\Sigma_1'', \Sigma_2'' : \beta' \to \beta_1$ are weaves obtained by finding a close deletion pair in β' and applying either weave from Lemma 4.10, followed by a sequence of braid moves.

Note that $\Sigma_1' \sim \Sigma_2'$, so it is enough to check that $\Sigma_1'' \sim \Sigma_2''$. Since Σ_1, Σ_2 have the same missing crossing in β_0, Σ_1'' and Σ_2'' have the same missing crossing in β'. Thus, Σ_1'' and Σ_2'' use the same close deletion pair in β', so the result now follows from Lemma 4.10. $\square$

Remark 4.12. Although it is natural to consider the above injection and missing crossings for more general weaves, these notions are not invariant under the equivalence relation. Indeed, one can check that the two paths in the 5-cycle for 1121 yield two different injections on crossings (with the same image), and the different paths for 12121 have different missing vertices. $\square$

For a positive braid word β, we define the *mutation graph* of β to be a graph with vertices given by the equivalence classes of Demazure weaves $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta\Delta, \Delta)$, and edges corresponding to mutations. As we discuss below, the equivalence classes of such weaves are expected to correspond to clusters of a certain cluster algebra, and (weave) mutations to correspond to (cluster) mutations. Thus, the mutation graph of β is expected to correspond to the exchange graph for this cluster algebra. Note that, by Theorem 4.6(b), any two equivalence classes of Demazure weaves in $\mathrm{Hom}_{\mathfrak{W}_n}(\beta\Delta, \Delta)$ are related by mutations. Let us formulate the following

Conjecture 4.13. *Suppose we oriented each mutation in the direction $(ss)s \to s(ss)$. For any positive braid β, this orientation descends on the mutation graph of β. With this orientation, the mutation graph has no oriented cycles.* $\square$

The conjecture is motivated by [13], where a similar statement was proven for the exchange graphs for quivers and cluster algebras; see also [14].

4.5. **Examples.** Let us study two explicit examples in detail, illustrating the material and results presented above.

Example 4.14. 2-strand braids. A braid on two strands is an element of Br_2: we denote by σ the unique Artin generator of this group, and by s the corresponding Coxeter generator $(12) \in S_2$. Each positive braid $\beta \in \mathrm{Br}_2$ has a unique braid word, which has the form σ^l, $l \geq 0$, and note that $\Delta = \sigma$. By abuse of notation, we will also write this word as s^l. We refer to the braid σ^l as the $(2,l)$-*torus braid*, since its (rainbow) closure is the $(2,l)$-torus link.

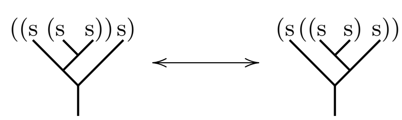
We have no braid moves in Br_2, so each weave $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(\beta, \beta')$ contains only trivalent vertices, cups and caps (and no 6- or 4-valent vertices). Each Demazure weave contains only trivalent vertices. As a planar trivalent graph, it is an acyclic graph, and so it is a disjoint union of binary trees. Each Demazure weave $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(\beta \cdot \Delta, \Delta)$ is naturally a rooted binary tree. By construction, all such binary trees with $l(\beta) + 1$ leaves are mutualy non-equivalent, but they are all related by mutations. If we orient each mutation $(ss)s \to s(ss)$, the oriented mutation graph will coincide with the classical Hasse graph of the Tamari lattice. It is known to be the 1-skeleton of a combinatorial polytope: the $(l(\beta) - 1)$-dimensional associahedron, see e.g. [79]. To summarize, we have the following

Lemma 4.15. *The mutation graph of the* $(2,l)$ *torus braid is the* 1*-skeleton of the* $(l-1)$*-dimensional associahedron.* □

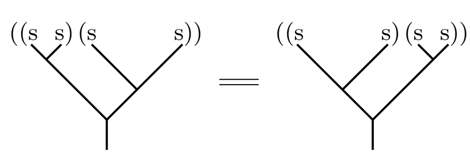
We can also understand each Demazure weave $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(s^l \cdot \Delta, \Delta)$ as a sequence of openings of crossings in the braid $s^l \cdot \Delta = s^{l+1}$. As we understand trivalent vertices $ss \to s$ as openings of the left crossing, Σ is actually a sequence of openings of crossings in β; the only crossing of Δ is the crossing of the concave end of Σ. Naturally, the sequence of crossings being opened can be seen as a permutation in S_l. The Tamari lattice is known to be both a sublattice and a lattice quotient of the weak order on permutations, see [79]. Note that a permutation is the same as a maximal chain in the Boolean lattice $2^{[l]}$ of the subsets of the set of crossing of β.

Finally, another elegant way to look at Demazure weaves $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(s^l \cdot \Delta, \Delta)$ is to consider them as monotone paths along the edges in the $l-$dimensional cube, with $2-$dimensional faces representing elementary moves (equivalences or mutations) between weaves. We illustrate this on the example of the $(2,3)$-torus braid $\beta = sss$ in Figure 7. Each edge of the cube is oriented downwards and corresponds to one trivalent vertex in a weave. Equivalently, it corresponds to opening a single crossing in β. Each vertex represents a horizontal cross-section away from the vertices of a Demazure weave $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(ssss, s)$; equivalently, it corresponds to a braid word obtained from β by the opening of some crossings. The underlined letters represent crossings that have been opened. For each edge of the weave in a horizontal slice, we can trace back its parents in $ssss$; these parents are in parentheses. The cube has the unique top vertex representing the braid $ssss$, and the unique bottom vertex representing s. Each Demazure weave can be seen as a monotone path along the edges from the top vertex to the bottom vertex.

The yellow face in the cube in Figure 7 illustrates the weave mutation given by



The blue face is the only face that does not represent a mutation. Two monotone paths related by the flip in this face correspond to two different possibilities to draw the same weave in such a way that each horizontal cross-section contains at most one trivalent vertex:



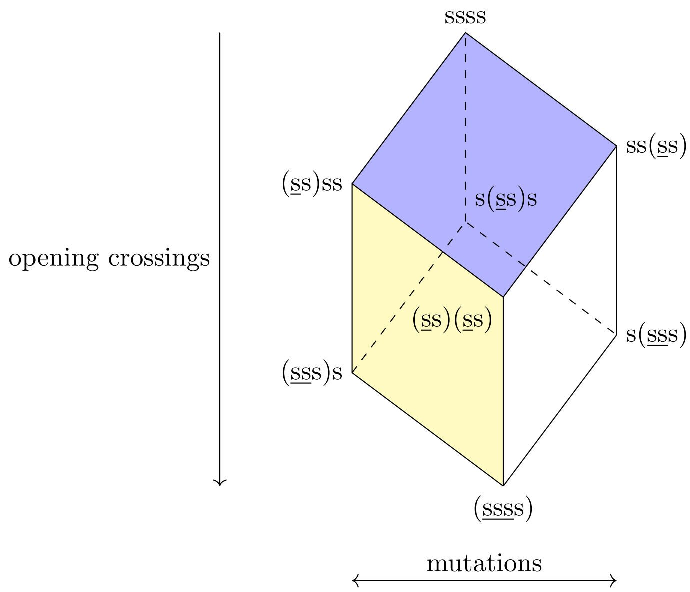


FIGURE 7. The Hasse graph of the Boolean lattice $2^{[3]}$. The top vertex is the initial braid word $\beta \cdot \Delta = s^3 \cdot s = s^4$, the bottom vertex represents $\Delta = s$. Demazure weaves $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_2}(ssss, s)$ correspond to monotone paths from the top vertex to the bottom vertex.

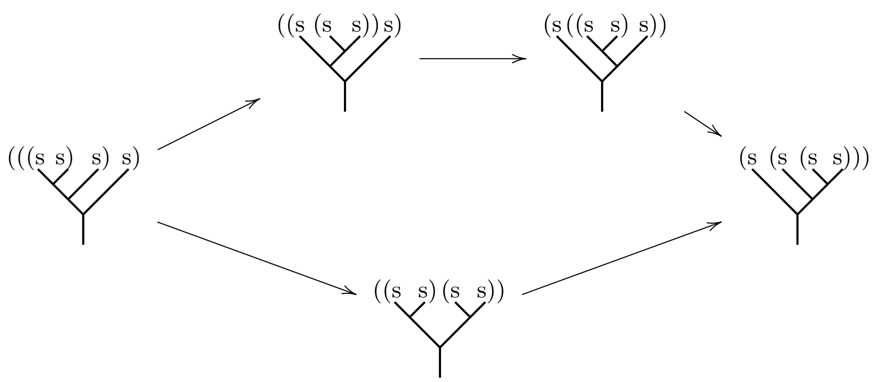


FIGURE 8. The mutation graph of the $(2, 3)$ torus braid sss. All mutations are oriented in the direction $(ss)s \to s(ss)$. It coincides with the Hasse graph of the Tamari lattice.

Two weaves are related by a single mutation if they are related by a polyonal flip in a non-blue face. In Figure7, mutations $(ss)s \to s(ss)$ correspond to replacements of two "left" sides of a square by its two "right" sides. The mutation graph is the 1-skeleton of 2-dimensional associahedron, that is, a pentagon. It is drawn on Figure 8 as the Hasse graph of the Tamari lattice of rooted binary trees with 4 leaves.

This concludes this example, focused on 2-stranded braids. The study of n-stranded braids and their weaves is, in general, much more elaborate (and thus interesting as well). This is illustrated in the next example. □

Example 4.16. The $(3, 2)$ **torus braid**. Consider the $(3, 2)$ torus braid $\beta = \sigma_1\sigma_2\sigma_1\sigma_2 = 1212$. Figure 9 illustrates Demazure weaves $1212 \cdot \Delta = 1212121 \to 212$ and relations between them. In Figure 9, we allow weaves with trivalent vertices $11 \to 1, 22 \to 2$ and 6-valent vertices represeting braid moves only in one direction: $121 \to 212$. Edges of the graph in Figure 9 represent single moves. We assume that each weave is drawn in such a way that each horizontal cross-section contains at most one vertex. Each vertex on Figure 9 represents a horizontal cross-section without vertices of an (a priori, not unique) weave. All edges are oriented downward. The weaves then correspond to monotone paths from the top vertex to the bottom vertex on the figure.

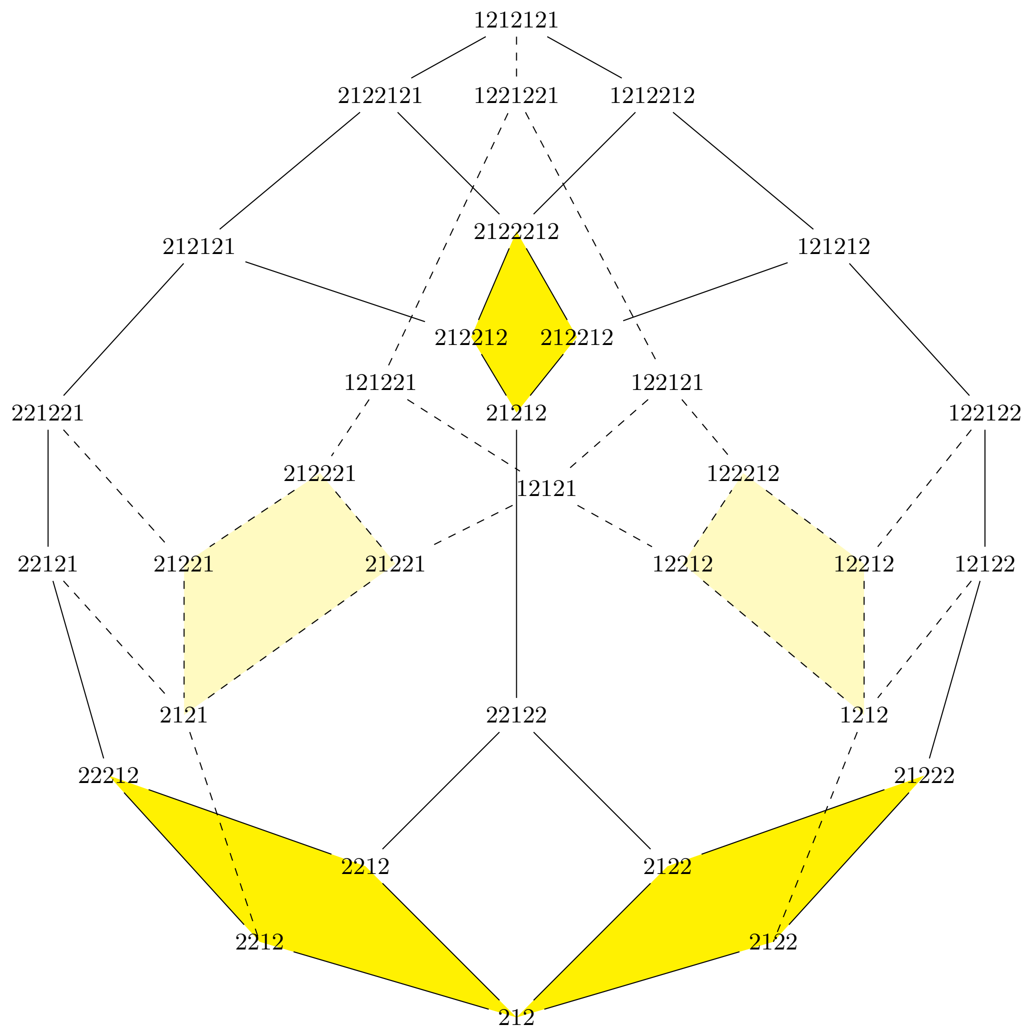


FIGURE 9. The top vertex is the initial braid word $\beta \cdot \Delta = s_1 s_2 s_1 s_2 \cdot s_1 s_2 s_1$, the bottom vertex represents $s_2 s_1 s_2$. Demazure weaves $\beta \cdot \Delta \to \Delta$ with only 6-valent vertices $s_1 s_2 s_1 \to s_2 s_1 s_2$ and 3-valent vertices allowed correspond to monotone paths from the top vertex to the bottom vertex.

It appears that there is a way to draw the graph as a 1−skeleton of a 3−dimensional polytope with 21 facets, although we did not try to find an explicit polytopal realization. The 2-dimensional (polygonal) faces correspond to the elementary moves between the paths. All the 2-dimensional faces are 4- or 8-gons:

(1) Yellow quadrilaterals correspond to mutations between the pairs of paths from sss to s.
(2) Other quadrilatelars correspond to some mutually independent distant operations. They represent two different possibilities to draw the same weave in such a way that each horizontal cross-section contains at most one trivalent vertex.
(3) Octagons correspond to the outer octagons in Section 4.2.5, they are formed by paths (A) and (E). Note that the inner vertices and paths do not appear since the moves $212 \to 121$ are not allowed.

We have no words containing 1121 or 1211 in our example, so the pentagons from 4.2.4 do not appear. In order to cover all Demazure weaves, we should allow the moves $212 \to 121$. In Figure 9, we should then replace each octagonal face by 5 faces from 4.2.5.

Two weaves are equivalent if the corresponding paths are separated by several white faces, and related by a single mutation if they are separated by one yellow face and several white faces. If we start from some monotone path, go by 2-dimensional faces and make the 360 degrees turn around the vertical axis, we go by all edges of the mutation graph of equivalence classes of paths exactly once. The mutation graph is a pentagon.

Each Demazure weave $\Sigma : 1212121 = \beta \cdot \Delta \to \Delta = 121$ is equivalent to a Demazure weave Σ' : $\beta \cdot \Delta \to 212$ concatenated with a single $6-$valent vertex $212 \to 121$. Indeed, if the last vertex in Σ is a $6-$valent vertex $\beta_{\ell(\Sigma)-1}(\Sigma) = 212 \to 121 = \beta_{\ell(\Sigma)}(\Sigma)$, then we define Σ' to be Σ with this vertex being removed. By construction, Σ is then the concatenation of Σ' with this vertex $\beta_{\ell(\Sigma)-1}(\Sigma) \to \beta_{\ell(\Sigma)}(\Sigma)$. Otherwise, we define Σ' to be Σ concatenated with a vertex $121 \to 212$. Then Σ is equivalent to Σ' concatenated with a vertex $212 \to 121$ via a cancellation move from Section 4.2.2. These arguments show that the mutation graph of Demazure weaves $\beta \cdot \Delta \to \Delta$ is isomorphic to the mutation graph of Demazure weaves $\beta \cdot \Delta \to 212$. Thus, the former, i.e. the mutation graph of β, is also a pentagon.

The appearance of the pentagon is not completely unexpected. Indeed, it coincides with the mutation graph of the torus braid $(2, 3)$. As we will see in Section 6, simplifying weaves $\beta\cdot\Delta \to \Delta$, as well as their equivalences and mutations, are related to augmentation varieties $\mathrm{Aug}(\Lambda(\beta))$. Since $\Lambda(3, 2) = \Lambda(2, 3)$, the fact that the mutations graphs of Demazure weaves of these two braids are isomorphic to each other is to be expected. □

Let us conclude this subsection on examples with two conjectures. First, given the Legendrian link equivalence between the $(2, n)$- and $(n, 2)$-torus links, and Lemma 4.15, we state the following:

Conjecture 4.17. *For the $(n, 2)$ torus braid β, the mutation graph of Demazure weaves $\beta \cdot \Delta \to \Delta$ is the 1-skeleton of the $(n - 1)$-dimensional associahedron.* □

Our conjectural 3-dimensional polytope on Figure 9 is similar to polytopes from [67, Figure 1] where the vertices encode equivalence classes of reduced expressions of elements in the braid group and edges correspond to braid moves (also oriented from $s_i s_{i+1} s_i$ to $s_{i+1} s_i s_{i+1}$). Reduced expressions are considered to be equivalent if they are related by a sequence of moves $s_i s_j \to s_j s_i, |i - j| \geq 2$. This equivalence relation is trivial in our $3-$strand case. It would be interesting to construct such polytopes for other braids.

Remark 4.18. The polytopes in [67] are the Hasse graphs of *second higher Bruhat orders* introduced by Manin and Shekhtman [65, 66], see also [90]. Given an arbitrary braid β, we can consider a similar oriented graph D_β. First, we associate a vertex to the braid β. We draw edges corresponding to moves $ss \to s$, $s_i s_{i+1} s_i \to s_{i+1} s_i s_{i+1}$, and $s_i s_j \to s_j s_i, |i - j| \geq 2$. We then contract all edges corresponding to moves $s_i s_j \to s_j s_i, |i - j| \geq 2$. This defines a poset with covering relations defined by edges. An element of the poset is an equivalence class of positive braid words with Demazure product Δ, with a certain extra decoration. Words are considered to be equivalent if they are related by a sequence of moves $s_i s_j \to s_j s_i, |i - j| \geq 2$. The decoration can be understood in terms of subsets of the set of crossings of the braid β; however, it is nontrivial to give a precise definition because of the issue discussed in Remark 4.12. We can also define the decoration in a non-combinatorial way by using variables from Section 5.3.

If we forget the decoration, this poset becomes a poset on the set of words with Demazure product Δ. Its analogue for all expressions of Δ and covering relations $ss \to s$ replaced by $ss \to e$ was defined by Elias [29] as an extension of the second higher Bruhat order to necessarily reduced words. It was used in the proof of the main result of the work [29], which we translated to our language as Theorem 4.6. Our weaves thus resemble saturated chains in the secong higher Bruhat order, which in turn can be seen as elements of the third higher Bruhat order. However, our equivalence relations differ from the one considered by Manin and Schechtmann. Note also that Thomas [88] defined the 0th Bruhat order to be the Boolean lattice. As we discussed in Example 4.14, Demazure weaves in $\mathfrak{W}_2$ can be seen as maximal chains in the 0th Bruhat order. In the present article, we will not explore the link between weaves and the theory of higher Bruhat orders further.

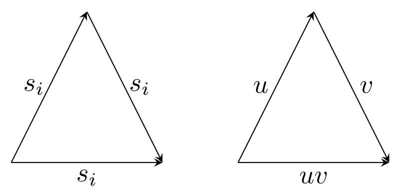
The graph D_β is not always a 1-skeleton of a polytope: e.g. D_{12122} is only a 1-skeleton of a union of two quadrilaterals. However, we have the following expectation:

Conjecture 4.19. *For an arbitrary positive braid word β, the poset complex of the oriented graph D_β is either a sphere or a ball. If it is a sphere, it admits a polytopal realization.* □

4.6. **Triangulations and weaves.** This subsection provides an interesting construction of weaves, by using certain labeled triangulations. It provides a systematic diagrammatic algorithm to produce exact Lagrangian fillings of the Legendrian link $\Lambda(\beta)$ which are *embedded* and, in general, non-Hamiltonian isotopic.

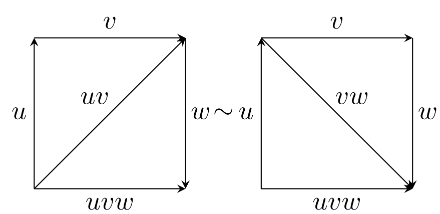
Given a positive braid word β, let us write the letters (crossings) of $\beta \cdot \Delta^2$ on the sides of a $(\ell(\beta) + 2\ell(w_0))-$gon. We will consider various labeled triangulations of this polygon. The easiest class of triangulations was defined by Mellit in [68, 69] as follows. We allow arbitrary many vertices inside the triangle, orient all edges and label them by the elements of S_n. If we change the orientation of an edge, we change the corresponding permutation to its inverse. We also require that all triangles have one of two types:

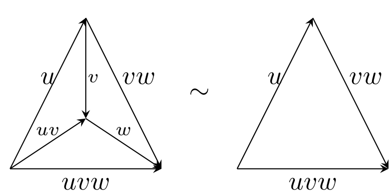
- All three sides are labeled by the same simple reflection s_i.
- The sides are labeled by permutations u, v and $u \cdot v$ such that $\ell(u \cdot v) = \ell(u) + \ell(v)$.



We will call such triangulations *admissible*. Given an admissible triangulation, we can algorithmically construct a weave, as follows. Choose a reduced expression for the permutation on every edge, then for triangles of the second type we can concatenate the reduced expressions for u and v and get a reduced expression for uv, which is possibly different from the one we chose. The two reduced expressions are related by a sequence of braid moves, which are translated to 6- and 4-valent vertices for weaves. The triangles of the first type corresponds to the trivalent vertices. Up to equivalence, this gives a well-defined weave, cf. Remark 4.8. Note that the weaves we obtain do not have caps or cups.

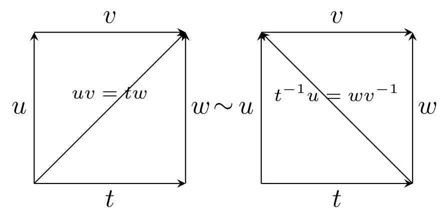
We can encode some of the moves between weaves in terms of triangulations. Given three permutations u, v and w such that $\ell(uvw) = \ell(u) + \ell(v) + \ell(w)$, we can make the following moves, which clearly do not change the weave:





For unlabeled triangulations, these are precisely the *Pachner moves* (also known as *bistellar flips*) in dimension 2. The result of Pachner [74] states that all triangulations of a polygon are related by such moves (the general version of this result holds for piecewise linear manifolds and bistellar flips in higher dimensions).

If we have four permutations u, v, w, t such that $uv = tw$, then $t^{-1}u = wv^{-1}$. Assuming that all these products are reduced, we have a move



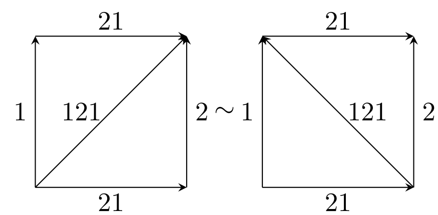
Note that in this case we get the equations

$$\ell(u) + \ell(v) = \ell(t) + \ell(w),\ \ \ell(u) + \ell(t) = \ell(v) + \ell(w)$$

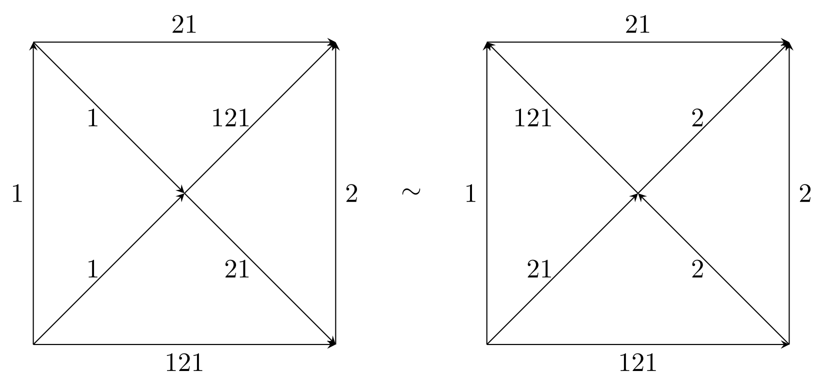
which imply

$$\ell(u) = \ell(w),\ \ \ell(v) = \ell(t).$$

Example 4.20. *The weave corresponding to either of the following diagrams is a single 6-valent vertex. The choice of a reduced expression (*121 *or* 212*) on the diagonal determines the triangle containing this 6-valent vertex.*



Finally, we can encode the 1212-move from Section 4.2.4 as the following move between triangulations:



Conversely, given a weave, we can consider the dual planar graph. It has triangular regions corresponding to 3-valent vertices in the weave, hexagonal regions corresponding to 6-valent vertices, and quadrilateral regions corresponding to 4-valent vertices. By choosing any admissible triangulation of each hexagon and quadrilateral, we get a triangulation of the whole polygon. The choice of the triangulation does not matter - for example, for the hexagon with sides labeled 1,2,1,2,1,2 there are 14 triangulations and 12 of them (those that do not contain triangles formed by three diagonals) are admissible. It is easy to see (using Example 4.20) that any two of them can be related by a sequence of the above moves, and correspond to the equivalent weaves.

Remark 4.21. In conclusion, we get a map from weaves to triangulations and from triangulations to weaves, but neither of them is a bijection. For future work, it would be interesting to find a complete set of moves between triangulations such that the corresponding equivalence classes are in bijection with the equivalence classes of weaves. □

4.7. **Demazure triangulations.** The correspondence between weaves and triangulations in Subsection 4.6 above is clear combinatorially. Nevertheless, it has a disadvantage: given a triangulation, it is unclear if the corresponding weave is simplifying or Demazure or, geometrically, if the corresponding Lagrangian surface is embedded in $\mathbb{R}^4$ (instead of merely immersed). In order to resolve this issue, we now introduce a special class of triangulations – with slightly more general labeling rules – which we refer to as *Demazure* triangulations.

Let us fix a reduced expression Δ of the longest element w_0 as in Example 2.2. Given a braid β, we consider an $(\ell(\beta)+\binom{n}{2}+1)$-gon, and we label the vertices with integers from 0 to $N := r+\binom{n}{2}$. We label the first $\ell(\beta)+\binom{n}{2}$ sides clockwise by the letters of $\beta\Delta$, and we label the last side (connecting 0 and N) by w_0. Given a diagonal in this polygon, we can label it by $\delta(\beta')$ where β' is the piece of β enclosed by this diagonal and δ is the Demazure product. By definition, a *Demazure* triangulation for a braid β is any triangulation of the $(\ell(\beta)+\binom{n}{2}+1)$-gon with sides and diagonals labeled with the above rules. A *Demazure triangle* is a triangle (with labeled sides) in a Demazure triangulation.

We can consider all possible triangulations of this polygon by diagonals (that is, there are no internal vertices). For any triangle in this picture, its three edges are labeled by u, v and $u \star v$ for some u and v. Note that the diagonals are labeled by elements of the symmetric group and not by braid words.

Definition 4.22. Let $\triangle$ be a triangle with sides u, v and $u \star v$. The *defect* of $\triangle$ is $\mathrm{def}(\triangle) = \ell(u)+\ell(v)-\ell(u\star v)$. $\square$

The following fact is rather immediate:

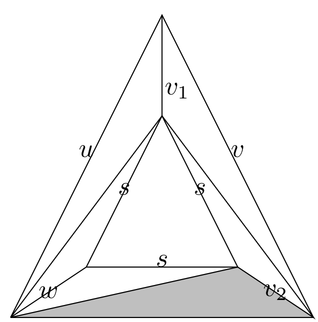
Lemma 4.23. *In a Demazure triangulation for β, we must have $\sum_{\triangle}\mathrm{def}(\triangle) = r$, where $r = \ell(\beta)$ is the length of the braid word β.*

Proof. We prove a more general statement: suppose that a diagonal (or the side with vertices 0 and N) encloses a braid β' with k crossings, and carries the label $u = \delta(\beta')$. Then, we will show that the sum of defects of the triangles above this diagonal equals $k-\ell(u)$. The statement of the Lemma follows by setting $u = w_0$ and then we get $r+\ell(w_0)-\ell(w_0) = r$. To prove this general statement, we use induction in $k \in \mathbb{N}$. Consider the triangle adjacent to the diagonal with u, its other sides are labeled v and w such that $v \star w = u$. By the assumption of induction, sum of defects above v equals $k_1-\ell(v)$ and the sum of defects above w equals $k_2-\ell(w)$, so the total sum of defects equals $k_1-\ell(v)+k_2-\ell(w)+\ell(v)+\ell(w)-\ell(u) = k-\ell(u)$. $\square$

Demazure triangulations relate to Subsection 4.6 as follows:

Proposition 4.24. *Any Demazure triangulation can be subdivided to an admissible triangulation.*

Proof. Consider a triangle with sides u, v and $u\star v = \delta(uv)$. If $u \star v = uv$, then this triangle is admissible. Otherwise, we use induction in $\ell(v)$. Let v_1 be the longest prefix of v such that uv_1 is reduced. Then we can write $v = v_1 s v_2$ such that $\ell(uv_1) = \ell(u)+\ell(v_1)$ but $\ell(uv_1 s) < \ell(u)+\ell(v_1)+1$. Therefore we can find a reduced expression w such that $uv_1 = ws$, and draw the following diagram:



The unmarked edges are labeled by $ws = uv_1, ws$ and sv_2. Now

$$u\star v = u\star v_1 \star s\star v_2 = w\star s\star s\star v_2 = ws\star v_2,$$

and by the assumption of induction we can subdivide the marked triangle with sides ws and v_2 into admissible ones. $\square$

The next result justifies the chosen nomenclature for a *Demazure* triangulation.

Corollary 4.25. *To each Demazure triangulation we can associate a Demazure weave* $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta \cdot \Delta,\ \Delta)$.

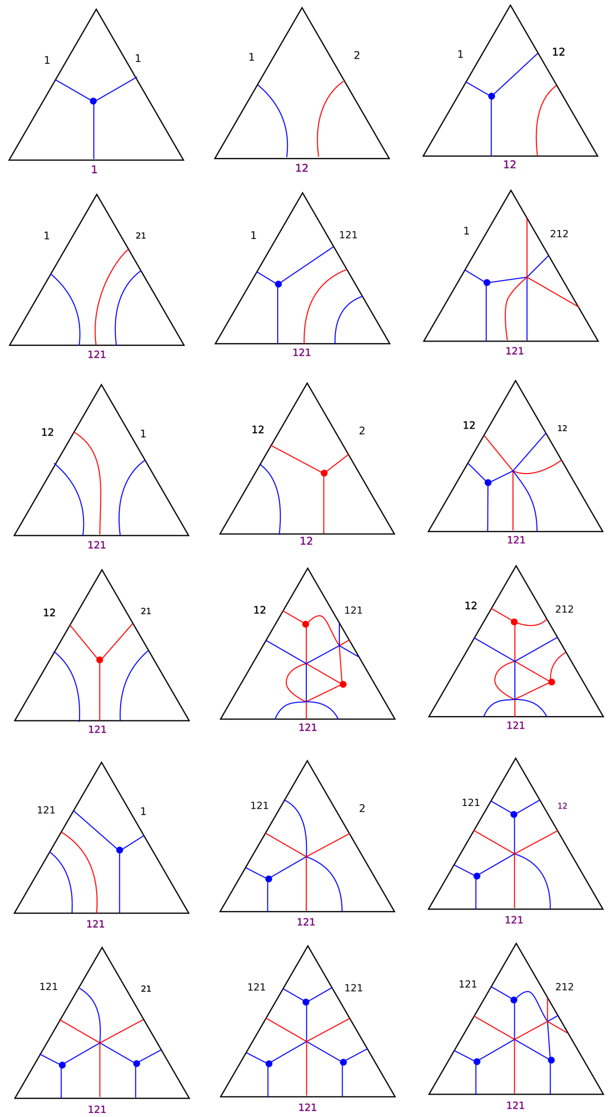


FIGURE 10. Possible algebraic weaves in triangles of a Demazure triangulation with only two colors. Note that fourth (i.e. the first in the second row) and seventh diagrams coincide as (pieces of) weaves; and the same is true for the sixth, ninth and fourteenth diagrams.

Proof. Given a Demazure triangulation, we can subdivide it to an admissible triangulation by Proposition 4.24. By results in Subsection 4.6, we can associate to this admissible triangulation a weave. This weave can be seen as a simplifying weave $\Sigma \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta \cdot \Delta, \Delta)$. Indeed, outside of the marked triangle it consists of several 6- and 4-valent vertices on the left encoding the relation $uv_1 = ws$, and a (correctly oriented) trivalent vertex at the center. Since we never use cups, this simplifying weave is Demazure. $\square$

In Figure 10 we depict possible pieces of algebraic weaves that can appear in triangles of a Demazure triangulation with only 2 colors.

Remark 4.26. From the perspective of contact geometry, embedded exact Lagrangian fillings for the (-1)-framed closure of β can also be constructed by expanding the Δ side to its $\binom{n}{2}$ sides. Thus the triangulation is that of an $(\ell(\beta) + 2\binom{n}{2})$-gon, instead of an $(\ell(\beta) + \binom{n}{2} + 1)$-gon. The Demazure labeling rule work in the same way: we start with sides labeled and complete the interior by always assigning the Demazure product to the remaining edge of a triangle. Figure 11 gives an example of these more general Demazure triangulation and a possible associated weave. $\square$

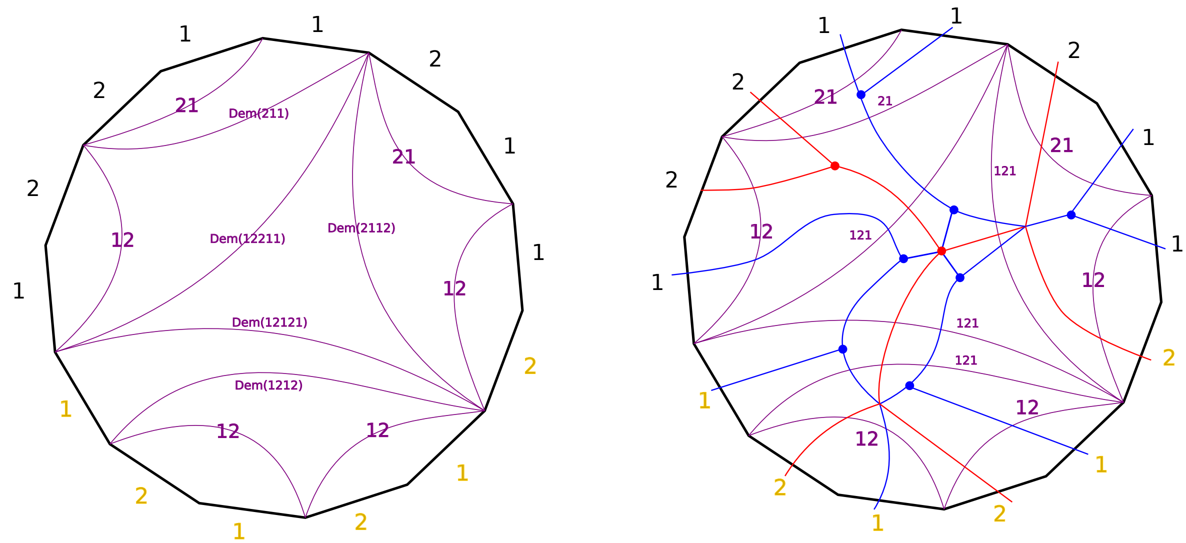


FIGURE 11. (Left) An extended Demazure triangulation as in Remark 4.26. (Right) The associated weave, which yields an embedded exact Lagrangian filling of the rainbow closure of $\beta = \sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1$.

Finally, note that there are several ways to fill a triangle by a weave (for example with sides 121,121,121), and these might be related by a mutation. Nevertheless, it follows from Theorem 4.11 that all ways to fill a triangle with defect 0 or 1 with a weave must be equivalent.

5. Algebraic Weaves, Morphisms, and Correspondences

This section develops the relative geometry of braid varieties, studying morphisms and correspondences between them. These correspondences are defined using weaves, and provide a functor from the category of algebraic weaves to the category of algebraic varieties and their correspondences.

5.1. **Correspondences.** Consider an algebraic weave Σ with braid words β_0 on the bottom and β_1 on the top. We now construct a correspondence between the two braid varieties $X_0(\beta_0), X_0(\beta_1)$. To each segment of an edge labeled by i we associate a variable z and the braid matrix $B_i(z)$; and the segments separated by a *vertex* or an *intersection point* with the yellow line carry different variables. In addition, each segment of a yellow line carries an undetermined upper uni-triangular matrix. All these variables and matrices can be considered as coordinates in the affine space $\mathbb{A}^\Sigma = \mathbb{C}^{\text{segments}} \times (\mathbb{C}^{\binom{n}{2}})^{\text{yellow segments}}$. The correspondence is then defined as follows:

Definition 5.1. Let Σ be a weave and $\tau \subseteq [1,2] \times \mathbb{R}$ a path on the plane transverse to Σ. The *monodromy* of the weave Σ along τ, also referred to as the monodromy of τ, is the ordered product of the following matrices:

(i) $B_i(z)$, if the path crosses an edge labeled by i with variable z *from left to right*,
(ii) $B_i(z)^{-1}$, if the path crosses an edge labeled by i with variable z *from right to left*,
(iii) U, if the path crosses a yellow line colored by U *downwards*,
(iv) U^{-1}, if the path crosses a yellow line colored by U *upwards*.

By definition, the correspondence variety $\mathcal{M}(\Sigma)$ associated to the weave Σ is the affine algebraic subvariety of $\mathbb{A}^\Sigma$ cut out by the conditions that the monodromy around a closed loop around every vertex (or intersection point with a yellow line) is trivial. $\square$

Note that Definition 5.1 implies that the monodromy around any closed loop is trivial. In particular, the monodromy around the loop encircling the whole weave with β_0 on the bottom and β_1 on the top equals $B(\beta_1)B(\beta_0)^{-1}\widetilde{U}^{-1}$, where $\widetilde{U}$ is the product of the upper-triangular matrices assigned to yellow lines on the left, so $B(\beta_1) = \widetilde{U}B(\beta_0)$. This implies the following:

Proposition 5.2. *Given $\pi \in S_n$, assume that $B(\beta_0)\pi$ is upper-triangular. Then $B(\beta_1)\pi$ is upper-triangular, and $\mathcal{M}(\Sigma)$ defines a correspondence $\mathcal{M}(\Sigma, \pi)$ between $X_0(\beta_0; \pi)$ and $X_0(\beta_1; \pi)$.*

Note that when $\pi = 1$ we obtain a correspondence $\mathcal{M}(\Sigma, 1)$ between the braid varieties $X_0(\beta_0)$ and $X_0(\beta_1)$. In general, $\mathcal{M}(\Sigma, 1)$ does not coincide with the variety $\mathcal{M}(\Sigma)$. Now, the composition of weaves is defined by vertical stacking, and corresponds to the following diagram:

$$\begin{tikzcd}
 & & \mathcal{M}(\Sigma,\pi) \arrow[dl] \arrow[dr] & & \\
 & \mathcal{M}(\Sigma_1,\pi) \arrow[dl] \arrow[dr] & & \mathcal{M}(\Sigma_2,\pi) \arrow[dl] \arrow[dr] & \\
X(\beta_0;\pi) & & X(\beta_1;\pi) & & X(\beta_2;\pi),
\end{tikzcd}$$

where Σ_1 is a weave from β_0 to β_1, Σ_2 is a weave from β_1 to β_2, Σ is a weave from β_0 to β_2 obtained by concatenation of Σ_1 and Σ_2, and it is easy to see that the middle square is Cartesian. In other words, $\mathcal{M}(\Sigma, \pi)$ is a convolution of correspondences $\mathcal{M}(\Sigma_1, \pi)$ and $\mathcal{M}(\Sigma_2, \pi)$. Also, it is clear that flipping a weave upside down corresponds to switching β_0 and β_1 and transposing the correspondence.

Therefore, to understand these correspondences, it is sufficient to describe them for the elementary weaves. These are described as follows:

(1) For s trivalent vertex colored by i, the correspondence $\mathcal{M}(\Sigma, \pi)$ embeds into $X(\beta_1; \pi)$ as the open locus $\{z_1 \neq 0\}$ and projects onto $X(\beta_0; \pi)$ with fibers $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. In terms of matrices, we have the identity
$$B_i(z_1)B_i(z_2) = \begin{pmatrix} -z_1^{-1} & 1 \\ 0 & z_1 \end{pmatrix} B_i(z_2 + z_1^{-1}).$$

(2) For 6-valent and 4-valent vertices, the corresponding character varieties $X(\beta_0; \pi)$ and $X(\beta_1; \pi)$ are isomorphic, and $\mathcal{M}(\Sigma, \pi)$ realizes this isomorphism. In terms of matrices, this corresponds to the identities
$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1),$$
$$B_i(z_1)B_j(z_2) = B_j(z_2)B_i(z_1) \ (|i - j| > 1).$$

(3) For a cup colored by i, the correspondence $\mathcal{M}(\Sigma, \pi)$ embeds into $X(\beta_1; \pi)$ as the closed locus $\{z_1 = 0\}$ and projects onto $X(\beta_0; \pi)$ with fibers $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$. In terms of matrices, we have the identity
$$B_i(0)B_i(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$
For a cap, we just use the transposed correspondence.

(4) The intersection of an edge and the yellow line corresponds to the identity
$$B_i(z)U = \widetilde{U}B_i(z')$$
from Lemma 2.13.

These four rules are justified by the following result:

Proposition 5.3. *In the construction of the correspondence variety $\mathcal{M}(\Sigma)$:*

(a) *The triangular matrices on yellow lines are uniquely determined by the variables on the edges.*
(b) *The output variables of each 3-, 6-, or 4-valent vertex are determined by the input variables.*

Proof. It follows from the proof of Lemma 2.13 that the equation $B_i(z)U = \widetilde{U}B_i(z')$ uniquely determines $\widetilde{U}$ and z' for given z and U. Therefore, for (a) it is sufficient to consider the yellow lines near trivalent vertices, cups, and caps.

For a 6-valent vertex, it is easy to see that $B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(w_1)B_i(w_2)B_{i+1}(w_3)$ implies $w_1 = z_3, w_2 = z_2 - z_1z_3, w_3 = z_1$, so the output variables are determined by the input ones. The proof for a 4-valent vertex is similar.

For a 3-valent vertex, we have an equation $B_i(z_1)B_i(z_2) = UB_i(w)$ which can be written as

$$\begin{pmatrix} 1 & z_2 \\ z_1 & 1+z_1z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & w \end{pmatrix} = \begin{pmatrix} b & a+bw \\ c & cw \end{pmatrix},$$

which implies $b = 1, c = z_1, w = (1 + z_1z_2)/c = z_2 + z_1^{-1}$ and $a = z_2 - bw = -z_1^{-1}$.

For a cup, we have $B_i(z_1)B_i(z_2) = U$ and similarly $z_1 = 0$ and U is determined by z_2. The case of a cap follows analogously. $\square$

By combining these facts, we obtain the following results:

Theorem 5.4. *Let Σ be a simplifying algebraic weave with m cups and r trivalent vertices. Then*

$$\mathcal{M}(\Sigma, \pi) \cong \mathbb{C}^m \times (\mathbb{C}^*)^r \times X_0(\beta_0; \pi),$$

the map to $X_0(\beta_0, \pi)$ is given by the projection to the third factor, and the map to $X_0(\beta_1; \pi)$ is injective. $\square$

Corollary 5.5. *Let Σ be a Demazure weave with r trivalent vertices. Then*

$$\mathcal{M}(\Sigma, \pi) = (\mathbb{C}^*)^r \times X(\beta_0; \pi),$$

and the map $\mathcal{M}(\Sigma, \pi) \to X(\beta_1; \pi)$ is an open embedding. $\square$

Corollary 5.5 follows since we have $\ell(\beta_0) + r = \ell(\beta_1)$, and thus $\mathcal{M}(\Sigma, \pi)$ and $X_0(\beta_1, \pi)$ have the same dimension. Let us now shift the focus to studying the relation between these correspondences and *opening* crossings of a positive braid; the latter having been a crucial ingredient in Section 2.1 and 3. The first observation, compare also with [37], is the following

Lemma 5.6. *Let β be a positive n-braid. For any $i \in [0, n-1]$, opening a crossing σ_i in the positive braid $\beta \cdot \Delta$ can be realized by the composition of the following three weaves:*

(a) *Move Δ next to σ_i and change the braid word for Δ to one which starts from σ_i. This only uses braid relations, or, equivalently, 6- and 4-valent vertices,*
(b) *Apply the trivalent vertex $\sigma_i\sigma_i \to \sigma_i$,*
(c) *Move Δ back to the end.*

Proof. By construction, the trivalent vertex $\sigma_i\sigma_i \to \sigma_i$ corresponds to opening the *left* crossing σ_i. It is easy to see that opening a crossing commutes with braid relations not involving this crossing, and the result follows. $\square$

Let us remark that the element Δ is *not* central in the braid group, and care is needed in Step (a) of Lemma 5.6: if $\beta = \beta_1\sigma_i\beta_2$ then the procedure in Lemma 5.6 is

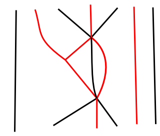
$$\beta_1\sigma_i\beta_2\Delta \to \beta_1\sigma_i\Delta\beta_2' \to \beta_1\sigma_i\Delta'\beta_2' \to \beta_1\Delta'\beta_2' \to \beta_1\Delta\beta_2' \to \beta_1\beta_2\Delta$$

where Δ' is an equivalent braid word to Δ that starts with σ_i, the opening of the crossing σ_i is performed in the third arrow, and all other arrows only involve braid moves or, equivalently, 4- and 6-valent vertices. Let us now give a concrete example of this procedure.

Example 5.7. *Suppose that $\beta = 1212$ and we want to open the second crossing in $\beta \cdot \Delta = 1212\underline{121}$. The above moves have the following form, where we have underlined Δ:*

$$1212\underline{121} = 121\underline{212}1 = 12\underline{121}21 \to 12\underline{221}21 \to 1\underline{212}21 \to 1\underline{121}21 = 11\underline{212}1 = 112\underline{121}.$$

The corresponding weave has the form



As a result, opening all crossings in a braid β, in some order, corresponds to a Demazure weave. Interestingly, the converse is also true, up to equivalence relation on weaves. This is the content of the following:

Theorem 5.8. *Let Σ be a Demazure weave. Then Σ is equivalent to a weave obtained by opening crossings in some order.*

Proof. Similarly to the proof of Theorem 4.11, any Demazure weave between braids β and β' such that $\ell(\beta) = \ell(\beta') + 1$ is equivalent to a weave corresponding to opening a crossing in β followed by some braid moves. Let us prove the statement of the theorem by induction on length. Given a weave from $\beta\Delta$ to Δ, choose a slice β' right below the first trivalent vertex. By the above argument the weave is equivalent to opening a crossing in β (which results in a braid $\beta''\Delta$) followed by some braid moves to β', and followed by the rest of the weave. By the assumption of induction, the weave from $\beta''\Delta$ to Δ is equivalent to opening crossings in β'' in some order. □

Finally, we now state the invariance of the correspondences $\mathcal{M}(\Sigma)$ under weave equivalence and deduce its main corollary, relating weave to the toric charts in Section 2. The invariance reads:

Theorem 5.9. *Let Σ_1, Σ_2 be equivalent Demazure weaves between β_0 and β_1, i.e. Σ_1, Σ_2 are related by a sequence of elementary moves (not mutations). Then, their associated correspondences $\mathcal{M}(\Sigma_1)$ and $\mathcal{M}(\Sigma_2)$ are isomorphic.*

Remark 5.10. It is shown in [19] that two Legendrian weaves related by an elementary move (or compositions of thereof) yield Hamiltonian isotopic Lagrangian projections, and also yield the same maps between the corresponding Legendrian Contact DGAs. Theorem 5.9 is an algebraic analogue of this statement. □

Theorem 5.9 will be proven momentarily, after we have deduced the following important consequence:

Corollary 5.11. *Let Σ be a Demazure weave between $\beta\Delta$ and Δ. Then the open chart $\mathcal{M}(\Sigma, w_0) \hookrightarrow X_0(\beta\Delta, w_0)$ coincides with one of the toric charts from Section 2.3.*

Proof. By Theorem 5.8 the weave Σ is equivalent to the weave Σ' obtained by opening crossings in some order. By Theorem 5.9 the open charts in $X_0(\beta\Delta, w_0)$ corresponding to Σ and Σ' coincide. □

As a short aside, we could be following [19] and have also defined the following correspondence $\mathcal{M}_{\mathrm{OBS}}(\Sigma)$, called the *flag moduli space* in [19]. This flag moduli is defined as follows. To each region of $([1,2]\times\mathbb{R})\setminus\Sigma$ we associate a flag in $\mathbb{C}^n$, if Σ goes between n-braids, and two regions separated by a line colored by i have flags in position s_i. The flags separated by a yellow line are required to coincide. There are two natural projections $\mathcal{M}_{\mathrm{OBS}}(\Sigma) \to \mathrm{OBS}(\beta_0), \mathcal{M}_{\mathrm{OBS}}(\Sigma) \to \mathrm{OBS}(\beta_1)$, so that $\mathcal{M}_{\mathrm{OBS}}(\Sigma)$ is a correspondence between $\mathrm{OBS}(\beta_0)$ and $\mathrm{OBS}(\beta_1)$. We can also define $\mathcal{M}_{\mathrm{OBS}'}(\Sigma) \subseteq \mathcal{M}_{\mathrm{OBS}}(\Sigma)$ as the closed subvariety given by the extra condition that the flag corresponding to the unbounded region on the far left of the weave coincides with the flag corresponding to the unbounded region on the far right. The variety $\mathcal{M}_{\mathrm{OBS}'}(\Sigma)$ is a correspondence between $\mathrm{OBS}'(\beta_0)$ and $\mathrm{OBS}'(\beta_1)$. In this setting, in line with Theorem 2.34, we can conclude the following:

Proposition 5.12. *Let $G = \mathrm{GL}(n)$ and $\mathcal{B} \subseteq G$ the Borel subgroup of upper-triangular matrices. There is a free action of $\mathcal{B}$ on $G \times \mathcal{M}(\Sigma)$, this action preserves $G \times \mathcal{M}(\Sigma, 1)$ and moreover*

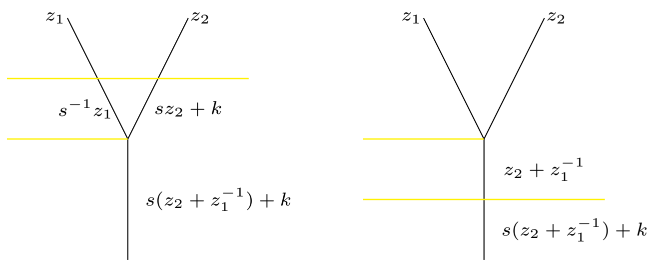
$$\mathcal{M}_{\mathrm{OBS}}(\Sigma) = (G \times \mathcal{M}(\Sigma))/\mathcal{B}, \qquad \mathcal{M}_{\mathrm{OBS}'}(\Sigma) = (G \times \mathcal{M}(\Sigma, 1))/\mathcal{B}.$$

Proof. An element in $G = \mathrm{GL}(n, \mathbb{C})$ corresponds to the choice of a basis in one of the regions on the plane. Given a point in $\mathcal{M}(\Sigma)$, we can define a basis in every other region, and the trivial monodromy condition ensures that this assignment is well defined. The flags in regions are induced by these bases. The action of $\mathcal{B}$ changes the basis in the rightmost region, but does not affect the flag in it.

Similarly to the proof of Theorem 2.34, we can propagate this action to the left and prove the desired isomorphism. □

5.2. **Proof of Theorem 5.9.** Let us prove Theorem 5.9. In order to do so, we directly check each elementary move from Section 4.2 separately. Cancellation of 4- and 6-valent vertices and commuting with distant colors are clear, and we do not include them in the list. Similarly. the various ways to resolve 12121 are related to each other by a sequence of 1212-moves, so it is sufficient to check the latter. The list of verifications needed for proving Theorem 5.9 now follows:

5.2.1. *Isotopies I: Changing the heights of vertices.* Changing the height of vertices does not change the the graph, but can change the yellow lines . Specifically, we need to understand how to slide yellow lines past 3-, 4- and 6-valent vertices.



The most interesting case is sliding through 3-valent vertex. In this case we have identity

$$\begin{pmatrix}0 & 1\\ 1 & z_1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & z_2\end{pmatrix}\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & z_1\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & \frac{b+cz_2}{a}\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & \frac{az_1}{c}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & \frac{b+cz_2}{a}\end{pmatrix}.$$

Therefore we have a transformation $(z_1,z_2)\to(s^{-1}z_1,sz_2+k)$ where $s=\frac{c}{a}$ and $k=\frac{b}{a}$. Note that under this transformation we have $z_2+z_1^{-1}\to s(z_2+z_1^{-1})+k$.

5.2.2. *Isotopies II: Cups and Caps.* Let us check that the two ways to define an upside down trivalent vertex in are equivalent. Indeed, the left picture corresponds to the changes of variables

$$(z)\to(0,u,z)\to(-u,z+u^{-1})$$

while the right picture corresponds to

$$(z)\to(z-w,0,w)\to((z-w)^{-1},w).$$

Here the cap on the left produces variables $(0,u)$ while the cap on the right produces variables $(0,w)$. We can idenify the two diagrams by setting $w=z+u^{-1}$, $u=-(z-w)^{-1}$.

Next, we compare two ways corresponding to the path $111\to 11\to\emptyset$ (see Section 4.2.1). The left one corresponds to changes of variables

$$(z_1,z_2,z_3)\to(z_2+z_1^{-1},z_3)\to\emptyset$$

and the cup is well defined if $z_2+z_1^{-1}=0$, that is, $1+z_1z_2=0$. The right diagram corresponds to

$$(z_1,z_2,z_3)\to(z_1+z_2^{-1},z_3+z_2^{-1})\to\emptyset$$

and is well defined if $z_1+z_2^{-1}=0$, which leads to the same equation. Note that $1+z_1z_2=0$ implies that both z_1 and z_2 are invertible, so that both trivalent vertices are well defined.

Finally, let us check the zigzag relation. On the left we have $(z)\to(0,0,z)\to(z)$ while on the right we have

$$(z)\to(z-u,0,u)\to(u)$$

which is well defined if $z-u=0$, so that $z=u$.

5.2.3. *The 1212-relation.* We refer to the notations in Section 4.2.4. The path $1212 \to 2122 \to 212 \to 121$ corresponds to changes of variables

$$(z_1, z_2, z_3, z_4) \to (z_3, z_2 - z_1z_3, z_1, z_4) \to (z_2, -z_2z_1^{-1} + z_3, z_4 + z_1^{-1}) \to (z_4 + z_1^{-1}, z_3 + z_2z_4, z_2)$$

Note that the second step corresponds to opening third crossing, which affects all other crossings. The other path $1212 \to 1121 \to 121$ corresponds to changes of variables

$$(z_1, z_2, z_3, z_4) \to (z_1, z_4, z_3 + z_2z_4, z_2) \to (z_4 + z_1^{-1}, z_3 + z_2z_4, z_2).$$

For completeness, we also include the computation for some of the other diagrams in Section 4.2.4. For 1121 we get two paths

$$(z_1, z_2, z_3, z_4) \to (z_2 + z_1^{-1}, z_3, z_4) \to (z_4, z_3 - z_4(z_2 + z_1^{-1}), z_2 + z_1^{-1})$$

and

$$(z_1, z_2, z_3, z_4) \to (z_1, z_4, z_3 - z_2z_4, z_2) \to (z_3 - z_2z_4, z_4 - z_1(z_3 - z_2z_4), z_1, z_2) \to$$
$$(z_4, -z_1^{-1}(z_4 - z_1(z_3 - z_2z_4)), z_2 + z_1^{-1})$$

Note that

$$-z_1^{-1}(z_4 - z_1(z_3 - z_2z_4)) = z_3 - z_4(z_2 + z_1^{-1}).$$

For 1211 we get two paths

$$(z_1, z_2, z_3, z_4) \to (z_2 - z_1z_3, z_3^{-1}z_2, z_4 + z_3^{-1}) \to$$
$$(z_4 + z_3^{-1}, z_3^{-1}z_2 - (z_4 + z_3^{-1})(z_2 - z_1z_3), z_2 - z_1z_3)$$

and

$$(z_1, z_2, z_3, z_4) \to (z_3, z_2 - z_1z_3, z_1, z_4) \to (z_3, z_4, z_1 - z_4(z_2 - z_1z_3), z_2 - z_1z_3) \to$$
$$(z_4 + z_3^{-1}, z_1 - z_4(z_2 - z_1z_3), z_2 - z_1z_3).$$

Note that

$$z_3^{-1}z_2 - (z_4 + z_3^{-1})(z_2 - z_1z_3) = z_3^{-1}z_2 - z_4z_2 + z_1z_3z_4 - z_3^{-1}z_2 + z_1 = z_1 - z_4(z_2 - z_1z_3).$$

The proof for other pair of adjacent colors is similar.

5.2.4. *The Zamolodchikov relation.* The left diagram in Section 4.2.6 represents the following path:

$$123121 \to 121321 \to 212321 \to 213231 \to 231213 \to 232123 \to 323123$$

which induces the following change of variables:

$$(z_1, z_2, z_3, z_4, z_5, z_6) \to (z_1, z_2, z_4, z_3, z_5, z_6) \to (z_4, z_2 - z_1z_4, z_1, z_3, z_5, z_6) \to$$
$$(z_4, z_2 - z_1z_4, z_5, z_3 - z_1z_5, z_1, z_6) \to (z_4, z_5, z_2 - z_1z_4, z_3 - z_1z_5, z_6, z_1) \to$$
$$(z_4, z_5, z_6, \widetilde{z}_3, z_2 - z_1z_4, z_1) \to (z_6, z_5 - z_4z_6, z_4, \widetilde{z}_3, z_2 - z_1z_4, z_1).$$

The right diagram represents the following path:

$$123121 \to 123212 \to 132312 \to 312132 \to 321232 \to 321323 \to 323123$$

which induces the following change of variables:

$$(z_1, z_2, z_3, z_4, z_5, z_6) \to (z_1, z_2, z_3, z_6, z_5 - z_4z_6, z_4) \to (z_1, z_6, z_3 - z_2z_6, z_2, z_5 - z_4z_6, z_4) \to$$
$$(z_6, z_1, z_3 - z_2z_6, z_5 - z_4z_6, z_2, z_4) \to (z_6, z_5 - z_4z_6, \widetilde{z}_3, z_1, z_2, z_4) \to$$
$$(z_6, z_5 - z_4z_6, \widetilde{z}_3, z_4, z_2 - z_1z_4, z_1) \to (z_6, z_5 - z_4z_6, z_4, \widetilde{z}_3, z_2 - z_1z_4, z_1).$$

Here $\widetilde{z}_3 = z_3 - z_1z_5 - z_2z_6 + z_1z_4z_6$.

This concludes the proof of Theorem 5.9, as required. Hence, we have established invariance of the correspondences $\mathcal{M}(\Sigma)$ under weave equivalence, showing that the functor is well-defined from the weave category.

5.3. **Mutation equivalence and rational maps.** The previous subsections have discussed weave equivalence thoroughly, let us now address weave mutations. First, note that any Demazure weave Σ from β_0 to β_1 defines a rational map Φ_Σ from $X_0(\beta_0,\pi)$ to $X_0(\beta_1,\pi)$, that is, the variables associated to crossings in β_1 can be expressed as rational functions in variables associated to crossings in β_0. This rational map Φ_Σ is defined on the image of $\mathcal{M}(\Sigma,\pi)$, but we can extend it to its maximal domain; we denote such extension by $\widehat{\Phi}_\Sigma$.

Example 5.13. *The weave $(ss)s \to ss$ corresponds to the rational map $(z_1,z_2,z_3)\mapsto(z_2+z_1^{-1},z_3)$, while the weave $s(ss)\to ss$ corresponds to the rational map $(z_1,z_2,z_3)\mapsto(-z_2-z_1z_2^2,z_3+z_2^{-1})$.*

Recall that two weaves are mutation equivalent if they are related by a sequence of equivalences and mutations. The maps associated to mutation equivalent weaves are related according to the following:

Theorem 5.14. *Let Σ,Σ' be two weaves which are mutation equivalent. Then, the corresponding maximal extensions of rational functions $\widehat{\Phi}_\Sigma$, $\widehat{\Phi}_{\Sigma'}$ coincide.*

Proof. By Theorem 5.9 the maps Φ_Σ and $\Phi_{\Sigma'}$ coincide for equivalent weaves even before mutations. Therefore it is sufficient to check mutations, using Example 5.13. One of the trivalent graphs involved in a mutation corresponds to the rational map

$$(z_1,z_2,z_3)\mapsto z_3+(z_2+z_1^{-1})^{-1}=z_3+\frac{z_1}{1+z_1z_2}$$

while the other corresponds to the rational map

$$(z_1,z_2,z_3)\mapsto z_3+z_2^{-1}+(-z_2-z_1z_2^2)^{-1}=z_3+\frac{1}{z_2}-\frac{1}{z_2(1+z_1z_2)}=z_3+\frac{z_1}{1+z_1z_2}.$$

Note that in the first case the map Φ_Σ is defined on the toric chart $\{z_1\neq 0, 1+z_1z_2\neq 0\}$ while in the second case it is defined on the chart $\{z_2\neq 0,1+z_1z_2\neq 0\}$, but in both cases it extends to the locus $\{1+z_1z_2\neq 0\}$ and the extensions agree. $\square$

Remark 5.15. Alternatively, we may state that the rational maps Φ_Σ and $\Phi_{\Sigma'}$ agree on the intersection of their corresponding domains, hence their maximal extensions must agree too. $\square$

5.4. **Torus actions and augmentation varieties.** In this subsection, given a simplifying weave Σ from β_1 to β_0, we will construct an action of the torus $T=(\mathbb{C}^*)^n/\mathbb{C}^*$ on the correspondence variety $\mathcal{M}(\Sigma)$ so that for every $\pi\in S_n$ both projections $\mathcal{M}(\Sigma,\pi)\to X_0(\beta_i;\pi)$, $i=0,1$, are T-equivariant. In particular, this allow us to define correspondence between augmentation varieties by Theorem 2.30.

First, we modify the action of T on $X_0(\beta;\pi)$ defined in Section 2.2 as follows. Take $\beta=\sigma_{i_1}\cdots\sigma_{i_\ell}\in\mathrm{Br}_n^+$ and let $w\in S_n$ be its corresponding permutation. We define an action of T on $\mathbb{C}^\ell$ by

$$\mathbf{t}.(z_1,\dots,z_\ell)=(d_1z_1,\dots,d_\ell z_\ell) \tag{5.1}$$

where $d_k=t_{w_k^\rho(i_k)}t^{-1}_{w_k^\rho(i_k+1)}$. Here, $w^\rho_{\ell-k}=s_{i_\ell}\cdots s_{i_{\ell-k+1}}=(s_{i_{\ell-k+1}}\cdots s_{i_\ell})^{-1}$. Thanks to (2.4) we have that $B_\beta(\mathbf{t}.z)=D^{-1}_{w^{-1}(\mathbf{t})}B_\beta(z)D_{\mathbf{t}}$, so for every permutation $\pi\in S_n$ we have an induced action on $X_0(\beta;\pi)$.

Remark 5.16. This torus action on $X_0(\beta;\pi)$ differs from the the action in Section 2.2 by conjugation by the permutation matrix w, thus the two actions are equivalent. The action used in Section 2.2 coincides with that considered in [68], while the action used in this section behaves better under morphisms given by weaves, as we will see below. $\square$

When we want to be explicit, we will write $\mathbf{t}\cdot_\beta(z_1,\dots,z_\ell)$ for the action (5.1). Note that this depends on the presentation of the braid β but it is readily verified that different presentations of β yield equivalent torus actions:

Lemma 5.17. *Let $\gamma_1,\gamma_2\in\mathrm{Br}_n^+$ and denote $r:=\ell(\gamma_1)$.*

(1) Let $\beta_1=\gamma_1\sigma_i\sigma_{i+1}\sigma_i\gamma_2$ and $\beta_0=\gamma_1\sigma_{i+1}\sigma_i\sigma_{i+1}\gamma_2$. Then, the map

$$f:\mathbb{C}^{\ell(\beta_1)}\to\mathbb{C}^{\ell(\beta_0)},\quad f(z)=(z_1,\dots,z_r,z_{r+3},z_{r+2}-z_{r+1}z_{r+2},z_{r+1},z_{r+4},\dots,z_\ell)$$

satisfies $f(\mathbf{t}\cdot_{\beta_1}z)=\mathbf{t}\cdot_{\beta_0}f(z)$.

(2) *Let* $\beta_1 = \gamma_1\sigma_i\sigma_j\gamma_2$ *and* $\beta_0 = \gamma_1\sigma_j\sigma_i\gamma_2$*, where* $|i-j|>1$*. Then, the map*

$$f : \mathbb{C}^{\ell(\beta_1)} \to \mathbb{C}^{\ell(\beta_0)}, \quad f(z) = (z_1, \dots, z_r, z_{r+2}, z_{r+1}, z_{r+3}, \cdots, z_\ell)$$

satisfies $f(\mathbf{t}\cdot_{\beta_1} z) = \mathbf{t}\cdot_{\beta_0} f(z)$. □

Let $D = \operatorname{diag}(a_1, \dots, a_n)$ be a diagonal matrix and $w \in S_n$, and let is write ${}^wD := \operatorname{diag}(a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)})$. The following lemma will be used to construct a (well-defined) torus action:

Lemma 5.18. *Let* w *be a permutation and* z *a variable of weight* $\mathbf{e}_{w(j)} - \mathbf{e}_{w(j+1)}$ *for some* $j = 1, \dots, n-1$*. Let* z_0 *be a variable of weight* $\mathbf{e}_{i+1} - \mathbf{e}_i$ *and define* z' *by the equation*

$$B_j(z)\,{}^wD_i(z_0) = {}^{ws_j}D_i(z_0)B_j(z'),$$

see (2.4)*. Then, the weight of* z' *is* $\mathtt{wt}(z') = \mathbf{e}_{s_iw(j)} - \mathbf{e}_{s_iw(j+1)}$.

Proof. First, note that z having weight $\mathbf{e}_{w(j)} - \mathbf{e}_{w(j+1)}$ is equivalent to saying that for any $\mathbf{t} \in T$:

$$\mathbf{t}.B_j(z) = D_{ws_j(\mathbf{t})}B_j(z)D^{-1}_{w(t)},$$

cf. Remark 2.20. Also, since $D_i(z_0) = \operatorname{diag}(1, \dots, -z_0^{-1}, z_0, 1, \dots, 1)$, where $-z_0^{-1}$ is in the i-th place, it is easy to see that $\mathbf{t}.{}^wD_i(z_0) = D_{w(\mathbf{t})}\,{}^wD_i(z_0)D^{-1}_{s_iw(\mathbf{t})}$. Now we compute

$$\begin{aligned}
\mathbf{t}.B_j(z') &= (\mathbf{t}.\,{}^{ws_j}D_i(z_0)^{-1})(\mathbf{t}.B_j(z))(\mathbf{t}.\,{}^wD_i(z_0)) \\
&= (D_{s_iws_j(\mathbf{t})}\,{}^{ws_j}D_i(z_0)^{-1}D^{-1}_{ws_j(\mathbf{t})})(D_{ws_j(\mathbf{t})}B_j(z)D^{-1}_{w(\mathbf{t})})(D_{w(\mathbf{t})}\,{}^wD_i(z_0)D^{-1}_{s_iw(\mathbf{t})}) \\
&= D_{s_iws_j(\mathbf{t})}B_j(z')D^{-1}_{s_iw(\mathbf{t})}
\end{aligned}$$

and the result follows. □

Finally, the desired statement regarding torus actions on our correspondences reads:

Proposition 5.19. *Let* Σ *be a simplifying algebraic weave from* β_1 *to* β_0*. Then, there is an action of an algebraic torus* T *on* $\mathcal{M}(\Sigma)$ *such that for every permutation* $\pi \in S_n$*:*

(1) T *preserves the correspondence variety* $\mathcal{M}(\Sigma, \pi)$*.*
(2) *The projections* $\mathcal{M}(\Sigma,\pi) \to X_0(\beta_1;\pi)$*,* $\mathcal{M}(\Sigma,\pi) \to X_0(\beta_0;\pi)$ *are equivariant.*

Proof. Thanks to Proposition 5.3 we have $\mathcal{M}(\Sigma) \subseteq \mathbb{C}^{\ell(\beta_1)}$, and we have an action of T on $\mathbb{C}^{\ell(\beta_1)}$ given by (5.1). Again by Proposition 5.3, this induces an action on $\mathcal{M}(\Sigma)$.

Note that, more generally, we have projections $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_1)}$, $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_0)}$. We will show that both of these maps are T-equivariant. This implies (1) and (2) above. By the definition of the T-action, the map $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_1)}$ is T-equivariant. To show that the map $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_0)}$ is T-equivariant, it suffices to do it for elementary weaves. For four and six-valent vertices, the result follows from Lemma 5.17 and Proposition 5.3.

Now we move on to three-valent vertices; we have $\beta_1 = \gamma_1\sigma_i\sigma_i\gamma_2$ and $\beta_0 = \gamma_1\sigma_i\gamma_2$. By Proposition 5.3 the map $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_0)}$ is given by $z \mapsto (z_1', \dots, z_r', z_{r+1} + z_r^{-1}, z_{r+2}, \dots, z_\ell)$, where $z_1', \dots, z_r'$ are determined by the equations

$$B_{i_{r-d}}(z_{r-d})U^d = U^{d+1}B_{i_{r-d}}(z'_{r-d}), \quad U^0 = U_i(z_r)D_i(z_r).$$

Note that the weights of $z_{r+2}, \dots, z_\ell$ are clearly preserved under the projection, so for simplicity we may assume that $\gamma_2 = 1$. We split this into a two-step process. We define $\widetilde{z}_1, \dots, \widetilde{z}_r$ via

$$B_{i_{r-d}}(z_{r-d})\widetilde{U}^d = \widetilde{U}^{d+1}B_{i_{r-d}}(\widetilde{z}_{r-d}), \quad \widetilde{U}^0 = U_i(z_r),$$

so that we have

$$B_{i_{r-d}}(\widetilde{z}_{r-d})\bar{U}^d = \bar{U}^{d+1}B_{i_{r-d}}(z'_{r-d}), \quad \bar{U}^0 = D_i(z_r).$$

Since $U_i(z_r)$ is uni-triangular, it follows from Lemma 2.22 that the T-weight of $\widetilde{z}_{r-d}$ coincides with that of z_{r-d} for $d = 0, \dots, r-1$. Now the result follows from Lemma 5.18.

Finally, we check cups: we have $\beta_1 = \gamma_1\sigma_i\sigma_i\gamma_2$ and $\beta_0 = \gamma_2$. The map $\mathcal{M}(\Sigma) \to \mathbb{C}^{\ell(\beta_0)}$ is given by $z \mapsto (z_1, \dots, z_r, z_{r+3}, \dots, z_\ell)$. Now, since $s_is_i = 1$, the result follows. □

Thanks to Proposition 5.19, we are able to define correspondences between certain augmentation varieties. Let β_0, β_1 be braid words, and let $\mathfrak{t}$ be a set of marked points on the strands $1, \ldots, n$ satisfying the following conditions:

(i) There is at most one marked point per strand and, by convention, it is placed to the right of all crossings in both β_0 and β_1,
(ii) Each component of both β_0 and β_1 contains at least one marked point.

For example, we can choose $\mathfrak{t} = \mathfrak{t}_s$ as in Section 2.6. We can then form the augmentation varieties $\mathrm{Aug}(\beta_1, \mathfrak{t})$ and $\mathrm{Aug}(\beta_0, \mathfrak{t})$. Now let $T_{\mathfrak{t}} \subseteq T$ be the torus defined by the equations $t_i = 1$ if the i-th strand has a marked point. Thanks to (a straightforward generalization of) Theorem 2.30, we have $\mathrm{Aug}(\beta_1, \mathfrak{t}) = X_0(\beta_1 \cdot \Delta; w_0)/T_{\mathfrak{t}}$ and $\mathrm{Aug}(\beta_0, \mathfrak{t}) = X_0(\beta_0 \cdot \Delta; w_0)/T_{\mathfrak{t}}$. In combination with the correspondences above, we then obtain the following result:

Corollary 5.20. *Let Σ be a simplifying algebraic weave from $\beta_1 \cdot \Delta$ to $\beta_0 \cdot \Delta$. Then, $T_{\mathfrak{t}}$ acts freely on $\mathcal{M}(\Sigma)$ and $\mathcal{M}(\Sigma, w_0)/T_{\mathfrak{t}}$ defines a correspondence between* $\mathrm{Aug}(\beta_1, \mathfrak{t})$ *and* $\mathrm{Aug}(\beta_0, \mathfrak{t})$*.*

5.5. **Weaves and stratifications.** In this concluding subsection, we explain how algebraic weaves can be used to stratify braid varieties; augmentation varieties can be similarly stratified. For that, recall that a simplifying weave Σ with a braid β_1 on the top and β_0 on the bottom defines an injective map

$$\mathcal{M}(\Sigma, \pi) : X_0(\beta_0; \pi) \times \mathbb{C}^a \times (\mathbb{C}^*)^b \hookrightarrow X_0(\beta_1; \pi),$$

where a is the number of cups and b is the number of trivalent vertices. Since each cup decreases the length by 2, and each trivalent vertex by 1, we get the equation $2a + b = \ell(\beta_0) - \ell(\beta_1)$.

We will be interested in simplifying weaves Σ with some braid γ on the top and the half twist Δ on the bottom. Since $X_0(\Delta; w_0)$ is a point, see Example 2.5, we obtain an injective map

$$\mathcal{M}(\Sigma, w_0) : \mathbb{C}^a \times (\mathbb{C}^*)^b \hookrightarrow X_0(\gamma; w_0), \ 2a + b = \ell(\gamma) - \binom{n}{2}.$$

By definition, we will say that a collection of simplifying weaves $\Sigma_1, \ldots, \Sigma_k$ stratifies the braid variety $X_0(\gamma, w_0)$ if the images of $\mathcal{M}(\Sigma_i)$ do not intersect each other and their union is $X_0(\gamma, w_0)$. At this stage, we can now conclude the following:

Theorem 5.21. (a) *Let γ be a positive braid word. Then the braid variety $X_0(\gamma, w_0)$ can be stratified by simplifying weaves from γ to Δ.*

(b) *Furthermore, given any Demazure weave Σ from γ to Δ, there is a stratification of $X_0(\gamma, w_0)$ by simplifying weaves where the correspondence*

$$\mathcal{M}(\Sigma) \cong (\mathbb{C}^*)^{\ell(\gamma) - \binom{n}{2}}$$

is the unique stratum of maximal dimension.

Proof. Let us first prove (a) by induction on $\ell(\gamma) \in \mathbb{N}$. If γ is reduced, then the matrix $B_\gamma(z_1, \ldots, z_{\ell(\gamma)})$ contains 1's corresponding to the permutation matrix for γ and independent variables elsewhere. Then $B_\gamma(z_1, \ldots, z_{\ell(\gamma)}) w_0$ contains $1's$ corresponding to the permutation matrix for γw_0, so it is never upper-triangular unless $\gamma w_0 = 1$. We conclude that $X_0(\gamma; w_0)$ is empty for $\gamma \neq \Delta$ and it is a point for $\gamma = \Delta$. In both cases the variety can be obviously stratified.

If γ is not reduced, then after applying some braid moves we get a braid with two crossings σ_i next to each other, let z_1 and z_2 be the corresponding variables. If $z_1 \neq 0$, we can apply a trivalent vertex and get a braid γ', and if $z_1 = 0$, we can apply a cup and get a braid γ''. By the assumption of induction, we can stratify $X_0(\gamma'; w_0)$ and $X_0(\gamma''; w_0)$ by simplifying weaves.

For (b), let us decompose Σ into elementary weaves: $\Sigma^{(1)}$ between $\gamma = \gamma^{(0)}$ and $\gamma^{(1)}$, $\Sigma^{(2)}$ between $\gamma^{(1)}$ and $\gamma^{(2)}$ etc. Clearly, we can decompose $X_0(\gamma)$ as follows:

$$X_0(\gamma; w_0) = \mathcal{M}(\Sigma) \sqcup \Big(X_0(\gamma; w_0) \setminus \mathrm{Im}\ \mathcal{M}(\Sigma^{(1)}) \Big) \sqcup \Big(\mathrm{Im}\ \mathcal{M}(\Sigma^{(1)}) \setminus \mathrm{Im}\ \mathcal{M}(\Sigma^{(1)}\Sigma^{(2)}) \Big) \sqcup \ldots$$

Let us prove that all these pieces can be further stratified by simplifying weaves. Indeed, if $\Sigma^{(i)}$ is a 6- or 4-valent vertex, then $\mathcal{M}(\Sigma^{(i)})$ is an isomorphism and

$$\mathrm{Im}\ \mathcal{M}(\Sigma^{(1)} \cdots \Sigma^{(i-1)}) = \mathrm{Im}\ \mathcal{M}(\Sigma^{(1)} \cdots \Sigma^{(i)}).$$

If $\Sigma^{(i)}$ is a trivalent vertex with variables z_1 and z_2 then

$$\mathrm{Im}\ \mathcal{M}(\Sigma^{(1)}\cdots\Sigma^{(i-1)})\setminus \mathrm{Im}\ \mathcal{M}(\Sigma^{(1)}\cdots\Sigma^{(i)})=\mathcal{M}(\Sigma^{(1)}\cdots\Sigma^{(i-1)})(W_i)$$

where W_i is the locus $\{z_1=0\}\subset X_0(\gamma^{(i-1)};w_0)$. In this case we can apply a cup to $\gamma^{(i-1)}$ and obtain a new braid $\widetilde{\gamma^{(i)}}$. Then W_i as an image of the correspondence for this cup, and by (a) we can stratify $X_0(\widetilde{\gamma^{(i)}};w_0)$ by simplifying weaves. □

Finally, we obtain the following consequence:

Corollary 5.22. *The braid variety $X_0(\gamma;w_0)$ is not empty if and only if γ contains some reduced expression for w_0 as a subword, or, equivalently, the Demazure product of γ equals w_0. In this case, $X_0(\gamma;w_0)$ is an irreducible complete intersection of dimension $\ell(\gamma)-\binom{n}{2}$.*

Proof. By [60, Lemma 3.4] a braid word γ contains some reduced expression for w_0 as a subword if and only if $\delta(\gamma)=w_0$. If $\delta(\gamma)=w_0$ then there is a Demazure weave from γ to w_0, so $X_0(\gamma;w_0)$ is not empty. By Theorem 5.21, if $X_0(\gamma;w_0)$ is not empty then there is a simplifying weave from γ to Δ, and γ contains some reduced expression for w_0 as a subword.

Since $X_0(\gamma;\pi)$ is cut out by $\binom{n}{2}$ equations in the affine space of dimension $\ell(\gamma)$, all its components have dimension at least $\ell(\gamma)-\binom{n}{2}$. On the other hand, if $\delta(\gamma)=w_0$ then by Theorem 5.21(b) the braid variety $X_0(\gamma;\pi)$ has unique stratum of dimension $\ell(\gamma)-\binom{n}{2}$ and all other strata have smaller dimension, therefore this variety is an irreducible complete intersection. □

Remark 5.23. In [68] A. Mellit proved that the complement to his toric chart from Section 2.5 can be decomposed into stata of the form $\mathbb{C}^a\times(\mathbb{C}^*)^b$ with $2a+b=\ell(\beta)$. Similarly to the proof of Theorem 2.28, one can check that these strata (originally defined in terms of Bruhat cells) can be realized by simplifying weaves. □

The stratifications we presented in Theorem 5.21 are far from unique. However, the number of strata of given dimension $a+b=\ell(\beta)-a$ does not depend of stratification. Indeed, if there are n_a such strata then the polynomial

$$\sum_a n_a q^a(q-1)^b=\sum_a n_a q^a(q-1)^{\ell(\beta)-2a}$$

counts points in the variety $X_0(\beta\Delta;w_0)$ over a finite field $\mathbb{F}_q$, or the coefficient at 1 in the standard basis expansion of the Hecke algebra element corresponding to β, or, equivalently, a coefficient in the HOMFLY-PT polynomial of β of lowest a-degree [55].

5.6. **From algebraic weaves to Legendrian weaves.** In the article [19], the first author and E. Zaslow associate to an algebraic weave a Legendrian surface in $\mathbb{R}^5$, and its (possibly immersed) Lagrangian projection to $\mathbb{R}^4$. The surface $L(\Sigma)$ is constructed via its wavefront $W(\Sigma)\subseteq\mathbb{R}^2\times\mathbb{R}$, which in turn is obtained by weaving n disjoint disks according to Σ, see [19, Definition 2.6]. The projection to $\mathbb{R}^2$ is a branched n-fold cover of the plane ramified at trivalent vertices, with the weave providing gluing conditions for the sheets of the cover. Also, we can think of the Lagrangian projection to $\mathbb{R}^4$ as an immersed Lagrangian cobordism between the Legendrain links obtained by the closure of the positive braids on the top and bottom of a weave.

Moreover, up to homeomorphism the surface $L(\Sigma)$ does not depend on the weave Σ, see [19, Theorem 1.1]. On the other hand, we can see from the proof of Lemma 3.3 that $\bar{\mathcal{S}}$ does not depend on Σ up to isotopy. So we may assume Σ is the weave obtained by opening the crossings in the order given by Theorem 2.28. The surface $\bar{\mathcal{S}}$ then can be obtained from the braid word β by following the procedure described in [68, Section 6.5]. It follows from the definitions that in this case $L(\Sigma)$ is homeomorphic to $\mathcal{S}$. The symplectic topology of $L(\Sigma)$ is more interesting, and provides one of the main motivation for our weave calculus. For instance, the following is proven in [19].

Theorem 5.24 ([19])**.** *Let Σ_1,Σ_2 be two algebraic weaves related by a sequence of elementary equivalence moves. Then the corresponding Lagrangian cobordisms $L(\Sigma_1),L(\Sigma_2)$ are Hamiltonian isotopic.*

The following simple geometric statement underscores the relevance of Demazure weaves, among all possible algebraic weaves:

Theorem 5.25. *Let Σ be an algebraic weave which is Demazure, i.e. with no cups or caps. Then the corresponding exact Lagrangian surface $L(\Sigma)$ is smoothly embedded in $\mathbb{R}^4$.*

Proof. Suppose that a braid β_1 is obtained from β by opening one crossing. Then by [27] there is an embedded Lagrangian saddle cobordism between the corresponding Legendrian links. Opening the crossings in a braid β in some sequence then gives an embedded Lagrangian cobordism as well.

By Theorem 5.8 any Demazure weave is related to the one corresponding to opening crossings by a sequence of elementary moves from Section 4.2 (no mutations). By Theorem 5.24 each elementary move corresponds to a Hamiltonian isotopy through embedded Lagrangians, so at every step we get an embedded Lagrangian. □

Remark 5.26. From a Floer-theoretic viewpoint, we conjecture that the correspondence $\mathcal{M}(\Sigma)$ induced between two augmentation varieties (of Legendrian links) is related to the augmentation variety for the Legendrian surface $\Lambda(\Sigma) \subset \mathbb{R}^5$. The latter augmentation variety is studied in the work of Y. Pan and D. Rutherford [76, 77]. □

Finally, we state the relationship that the Legendrian surface $L(\Sigma)$ has to the toric charts in the augmentation variety $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ where, recall, $\mathfrak{t}_c$ is a choice of marked points in β such that each component of the closure contains a unique marked point. Thanks to Lemma 5.6 and Corollary 5.20, any toric chart in $\mathrm{Aug}(\beta, \mathfrak{t}_c)$ is obtained by a simplifying weave Σ from $\beta \cdot \Delta^2$ to Δ^2 as in Section 5.5. Recall from Section 3.2 that each toric chart $\mathfrak{T}$ has the form $\mathfrak{T} = \mathrm{Spec}(\mathbb{Z}[H_1(\bar{\mathcal{S}}, \mathbb{Z})]) \times S$, where $\bar{\mathcal{S}}$ is a closed surface and S is a symplectic torus that depends only on the number of components of β. Moreover, $\omega|_{\mathrm{Spec}(\mathbb{Z}[H_1(\bar{\mathcal{S}},\mathbb{Z})])}$ is the intersection form on $H_1(\bar{\mathcal{S}}, \mathbb{Z})$. In summary, we conclude:

Theorem 5.27. *Let Σ be a simplifying weave from $\beta \cdot \Delta$ to Δ, and assume that the closure of the braid β has k components. Then, the weave Σ determines an algebraic torus $\mathfrak{T} \subseteq \mathrm{Aug}(\beta, \mathfrak{t}_c)$, and this torus is isomorphic to $\mathrm{Spec}(\mathbb{Z}[H_1(L(\Sigma), \mathbb{Z})]) \times S$, where S is a $2(k-1)$-dimensional symplectic torus. In addition, the restriction of the symplectic form $\omega|_{\mathfrak{T}}$ to $\mathrm{Spec}(\mathbb{Z}[H_1(L(\Sigma), \mathbb{Z})])$ coincides with the intersection form on $H_1(L(\Sigma, \mathbb{Z}))$.*

This concludes the proofs of the main results presented in the introduction. In the following section, we start exploring the relation between algebraic weaves and cluster algebras through a few examples.

6. Cluster Coordinates from Algebraic Weaves

Consider $\beta \in \mathrm{Br}_n^+$ and a n-weave $\Lambda : \Delta^2 \preceq \beta$, i.e. $\Lambda \in \mathrm{Hom}_{\mathfrak{W}_n}(\beta, \Delta^2)$, from Δ^2 at the concave end to β in its convex end. The Legendrian link $\Lambda(\Delta^2)$ is the standard Legendrian n-unlink, and thus it admits a unique embedded exact Lagrangian filling. In consequence, such a weave Λ defines a embedded exact Lagrangian filling $L(\Lambda)$. The associated augmentation

$$\varepsilon_{L(\Lambda)} : \mathcal{A}(\Lambda(\beta)) \longrightarrow \mathbb{Z}[H_1(L(\Lambda), \mathbb{Z})]$$

defines a $\mathbb{Z}$-torus $\mathrm{Spec}(\mathbb{Z}[H_1(L(\Lambda), \mathbb{Z})])$ inside the augmentation variety $\mathrm{Aug}(\Lambda(\beta))$ of $\Lambda(\beta)$, which complexifies to a toric chart $(\mathbb{C}^*)^{b_1(L(\Lambda))} \cong \mathrm{Spec}(\mathbb{C}[H_1(L(\Lambda), \mathbb{Z})]) \subseteq \mathrm{Aug}(\Lambda(\beta))$ when considered as algebraic complex varieties [20]. Complex points in this toric chart are to be understood geometrically as choices for a $\mathbb{C}^*$-local system on the *one* fixed Lagrangian filling $L(\Lambda)$, the toric coordinates specifying the local system. The goal of this section is to obtain *cluster* A-coordinates in such toric charts by studying (combinatorial avatars of) holomorphic strips with asymptotic ends in the Reeb chords of $\Lambda(\beta)$. Since cluster A-coordinates readily recover cluster X-coordinates, this yields the first known Floer-theoretical description of both cluster A- and X-coordinates in terms of the augmented values of Reeb chords.[15]

Remark 6.1. We remark that all the weaves $\Lambda : \Delta^2 \preceq \beta$ drawn in this section can also be drawn as weaves $\Sigma : \beta \cdot \Delta \to \Delta$ which give toric charts in $X_0(\beta \cdot \Delta; w_0)$, see Corollary 5.5, and therefore by Theorem 2.30 also on $\mathrm{Aug}(\Lambda(\beta))$. The link between these two perspectives is given by Theorem 5.27 above. □

[15]The present manuscript will describe the necessary holomorphic strips combinatorially, and the upcoming work [21] shows that these are indeed legitimate holomorphic strips.

6.1. **The Case of $n=2$ Weaves.** Consider $\beta = \sigma_1^{n+2} \in \mathrm{Br}_2^+$ and a weave $\Lambda : \Delta^2 \preceq \beta$. In this case, the Legendrian link $\Lambda(\Delta^2)$ is the standard Legendrian 2-unlink and the weave $\Lambda : \Delta^2 \preceq \beta$ defines a embedded exact Lagrangian filling $L(\Lambda)$ of the max-tb $(2,n)$-torus link $\Lambda(\beta) \subseteq (\mathbb{S}^3, \xi_{\mathrm{st}})$.

The first step is a computation of the rigid holomorphic strips

$$u : (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (\mathbb{S}^3 \times \mathbb{R}_t, d(e^t\alpha_{\mathrm{st}})), \quad \bar{\partial}_J u = 0, \quad u(\partial\mathbb{D}^2) \subseteq L(\Lambda)$$

with exactly *one* positive puncture at a Reeb chord ρ of $\Lambda(\beta)$. The reason these strips are crucial is that the variable $z_\rho \in \mathbb{C}$ associated to a Reeb chord ρ algebraically accounts for the (partial) holonomy along the boundary of these strips. It is proven in [19, Section 7] that the microlocal monodromies along absolute 1-cycles $\gamma \in H_1(L(\Lambda), \mathbb{Z})$ are cluster coordinates, and thus we aim at expressing the microlocal monodromies around γ as a function of the variables z_ρ associated to Reeb chords ρ of $\Lambda(\beta)$. This is done in three steps:

(i) Choosing an appropriate basis $\{\gamma_i\}$, $i \in [1, b_1(L(\Lambda))]$, for $H_1(L(\Lambda), \mathbb{Z})$ using the 2-weave,
(ii) Computing of the rigid holomorphic strips and their relative homology classes,
(iii) Express each term γ_i of the basis in (i) in terms of the relative homology classes of the holomorphic strips in (ii).

Remark 6.2. Throughout this subsection, we will use $\beta = \sigma_1^9$ as our running example, so that $\Lambda(\beta)$ is the max-tb Legendrian representative of the $(2,7)$-torus knot. Any interesting property of the general case already appears for such $\beta = \sigma_1^9$. Figure 12 illustrates the first two steps $(i), (ii)$ in this example, as we momentarily explain. □

The choice of basis in (i) is given by the following elementary, where we use the notation from [19]:

Lemma 6.3. *Let $\Lambda : \Delta^2 \preceq \beta$ be a vertical 2-weave and $\beta = \sigma_1^{n+2} \in \mathrm{Br}_2^+$. Suppose that the trivalent vertices are labeled $v_1, \ldots, v_{b_1(L)}, v_{b_1(L)+1}$ scanning from the top downwards. Then:*

(1) *For each v_i, $i \in [1, b_1(L)]$, there exists exactly one absolute 1-cycle $\gamma_i \subseteq L(\Lambda)$, $i \in [1, b_1(L)]$ which is represented by a short* I*-cycle starting at v_i.*

(2) *The set of $\{\gamma_i\}$, $i \in [1, b_1(L(\Lambda))]$, is basis of $H_1(L(\Lambda), \mathbb{Z})$ with intersection form*

$$\langle \gamma_i, \gamma_j \rangle = \begin{cases} 1 & \text{if } e(i) = e(j), \\ \pm 1 & \text{if } i = e(j), \\ 0 & \text{else.} \end{cases}$$

where $v_{e(i)}$ denotes the ending vertex of the short I*-cycle representing γ_i. If if $i = e(j)$, then $\langle \gamma_i, \gamma_j \rangle = 1$ if and only if the* I*-cycle representing γ_j is given by the edge at the upper-right of the trivalent vertex v_i. Equivalently, $\langle \gamma_i, \gamma_j \rangle = -1$ if and only if the* I*-cycle representing γ_j is given by the edge at the upper-left of v_i.* □

Note that Lemma 6.3 only holds for $n=2$ weaves, and Y-cycles are typically needed for $N \geq 3$. The basis $\{\gamma_i\}$ in Lemma 6.3 is depicted in Figure 12 for $\beta = \sigma_1^9$.

The classification of *rigid* holomorphic strips starting at the Reeb chords and with boundary on $L(\Lambda)$ is proven in [21], which geometrizes our previous work [17]. For the case at hand, the core technical result we need is the following:

Lemma 6.4 ([17, 21])**.** *Let $\Lambda : \Delta^2 \preceq \beta$ be a vertical 2-weave and $\beta = \sigma_1^{n+2} \in \mathrm{Br}_2^+$. Then*

(1) *For each Reeb chord ρ there exists a rigid holomorphic strip $T_i(\rho^+)$ with a positive puncture at ρ, no negative punctures, and ending at a trivalent vertex v_i if and only if there exists a continuous path in the complement of the weave $\mathbb{R}^2 \backslash G(\Lambda)$ which connects v_i and (the projection of) ρ.*

(2) *This rigid holomorphic strip $T_i(\rho^+)$ is unique and the relative cycle $\partial T_i(\rho^+) \subseteq (L(\Lambda), \Lambda(\beta))$ intersects the basis $\{\gamma_i\}$ according to*

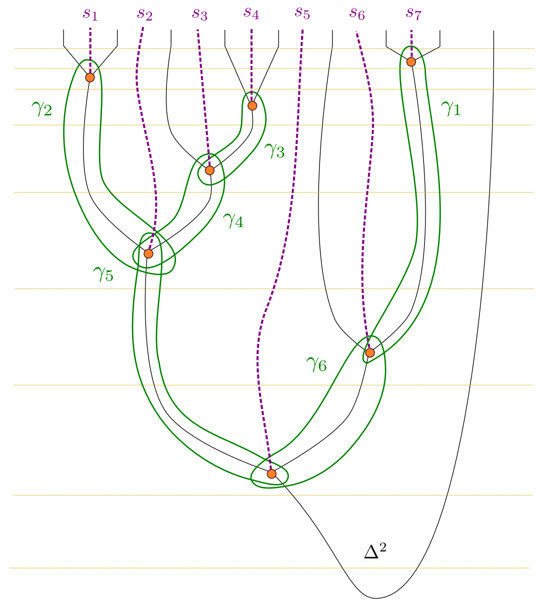


FIGURE 12. An algebraic 2-weave representing an embedded exact Lagrangian filling $L(\Lambda)$ of the max-tb $(2,7)$-torus knot $\Lambda(\sigma_1^9)$. Smoothly, $L(\Lambda)$ is a once-punctured genus-3 surface with $b_1(L(\Lambda)) = 6$. A basis $\{\gamma_i\}$ for $H_1(L(\Lambda),\mathbb{Z})$, represented by short I-cycles, is drawn in green. The dashed purple lines indicate the Morse flow tree trajectories associated to the holomorphic strips with asymptotic positive punctures at z_ρ. Note that s_ρ denotes the restriction of the variable z_ρ to its non-vanishing stratum $\{z_\rho \neq 0\}$ in the augmentation variety $\mathrm{Aug}(\Lambda(\sigma_1^9))$.

$$\langle \partial T_i(\rho^+), \gamma_j \rangle = \begin{cases} 1 & \textit{if } i = j, \\ -1 & \textit{if } i = e(j), \\ 0 & \textit{else.} \end{cases}$$

The Morse flow trees associated to the rigid holomorphic strips $T_i(\rho^+)$ in Lemma 6.4 are depicted in Figure 12 as dashed purple lines. The cohomological dual of the relative homology class $[\partial T_i(\rho^+)] \in H_1(L(\Lambda), \Lambda(\beta))$ of the boundary $\partial T_i(\rho^+)$ is denoted by s_i. We also identify the restriction of the complex variable $z_i \in \mathbb{C}$, associated to a Reeb chord ρ_i of $\Lambda(\beta)$, to its non-vanishing locus $\{z_i \neq 0\} \subseteq \mathrm{Aug}(\Lambda(\beta))$ with this s_i, that is, $s_i = z_i|_{\{z_i \neq 0\}}$.

The third stage is identifying the cluster coordinates, which are given by the microlocal monodromies along the γ_i absolute 1-cycles, in terms of the monodromies associated to the relative classes $[\partial T_i(\rho^+)]$. This thus becomes an homological problem: expressing each $\gamma_i \in H_1(L(\Lambda))$ in terms of $[\partial T_i(\rho^+)]$. This can be done using the intersection form in Lemma 6.4.(2), as an expression of Poincaré-Lefschetz duality. For simplicity, we will denote $s_i = [\partial T_i(\rho^+)]$, identifying cap intersections and dual pairing. A simple computation leads to the following identity:

$$\gamma_i = \prod_{\rho_j \in A(i)} s_j,$$

where $A(i)$ denotes the set of Reeb chords ρ_j of $\Lambda(\beta)$ which lie *above* the vertex v_i. That is, consider the binary tree starting at v_i growing upwards, then $\rho_j \in A(i)$ if and only if ρ_j is between the leftmost and rightmost upper edges of the tree.

Example 6.5. Consider the running example $\beta = \sigma_1^9$ as depicted in Figure 12. The expression of the absolute 1-cycles γ_i, represented by short I-edge, in terms of the s_i-variables is

$$\gamma_1 = s_7 = (s_1 s_2 s_3 s_4 s_5 s_6)^{-1}, \quad \gamma_2 = s_1, \quad \gamma_3 = s_4,$$

$$\gamma_4 = s_3 s_4, \quad \gamma_5 = s_1 s_2 s_3 s_4, \quad \gamma_6 = s_6 s_7 = (s_1 s_2 s_3 s_4 s_5)^{-1},$$

where we have used the homological relation $s_1s_2s_3s_4s_5s_6s_7 = 1$, which is the only existing relation, as $\Lambda(\sigma_1^9)$ is a knot, and not a link. The expression of the s_i-variables in terms of the z_i-variables is

$$z_1 = s_1, \quad z_2 = s_2 - \frac{1}{s_1} - \frac{1}{s_3}, \quad z_3 = s_3 - \frac{1}{s_4}, \quad z_4 = s_4, \quad z_5 = s_5 - \frac{1}{s_4} + \frac{1}{s_3s_4^2} - \frac{1}{s_2s_3^2s_4^2} - \frac{1}{s_6}, \quad z_6 = s_6 - \frac{1}{s_7},$$

$$z_7 = s_7, \quad z_8 = \frac{1}{s_6s_7^2} - \frac{1}{s_5s_6^2s_7^2} - \frac{1}{s_7}, \quad z_9 = s_2s_3s_4s_5s_6s_7 - s_2s_3^2s_4^2s_5^2s_6^2s_7^2 + s_5s_6^2s_7^2.$$

This follows from the fact that, from the perspective of decomposable Lagrangian cobordisms, the Lagrangian filling associated to Λ is obtained by opening the (contractible) crossings $z_1, \ldots, z_9$ in the order $(7, 1, 4, 3, 2, 6, 5)$. In consequence, the cluster A-coordinates $A_i := A(\gamma_i) = A(\gamma_i(s(z))) = A(z)$ in the toric chart $\mathrm{Spec}(\mathbb{C}[H_1(L(\Lambda), \mathbb{Z})]) \subseteq \mathrm{Aug}(\Lambda(\beta))$ associated to this Lagrangian filling $L(\Lambda)$ are

$$A_1 = z_7, \quad A_2 = z_1, \quad A_3 = z_4, \quad A_4 = z_3z_4+1, \quad A_5 = z_3z_4+z_1\left(z_4 + z_2\left(z_3z_4+1\right)\right)+1, \quad A_6 = 1+z_6z_7.$$

Since the z_i-variables related to the Plücker coordinates according to

$$P_{79} = B_1(z_7)_{2,2} = z_7, \quad P_{13} = B_1(z_1)_{2,2} = z_1, \quad P_{46} = B_1(z_4)_{2,2} = z_4, \quad P_{36} = (B_1(z_3)B_1(z_4))_{2,2} = z_3z_4+1,$$

$$P_{16} = (B_1(z_1)B_1(z_2)B_1(z_3)B_1(z_4))_{2,2} = z_1z_2+(z_1 + (z_1z_2+1)\,z_3)\,z_4+1, \quad P_{69} = (B_1(z_6)B_1(z_7))_{2,2} = z_6z_7+1.$$

we conclude that this cluster chart $C(\Lambda)$ is given by the cluster A-coordinates

$$\{P_{13}, P_{16}, P_{36}, P_{46}, P_{69}, P_{79}\}.$$

This corresponds exactly to the diagonals of the triangulation dual to the 2-weave, thus recovering the classical description of cluster A-coordinates in Type A via Floer-theoretic methods. □

Finally, we emphasize that there is an abundance of regular functions in $\mathcal{O}(\mathrm{Aug}(\Lambda(\beta)))$, the challenge, which we have thus far solved for 2-weaves, is deciding which regular functions are cluster coordinates and specify the subsets of them corresponding to a cluster chart. The above discussion explains how to obtain a cluster chart, with its cluster coordinates, in $\mathrm{Aug}(\Lambda(\beta))$ from an exact Lagrangian filling associated to a 2-weave, and this is done by using counting holomorphic strips and keeping track of their homology classes. Let us conclude our study of the $n = 2$ case by comparing two Legendrian 2-weaves, differing exactly by one mutation, and their cluster A-coordinates.

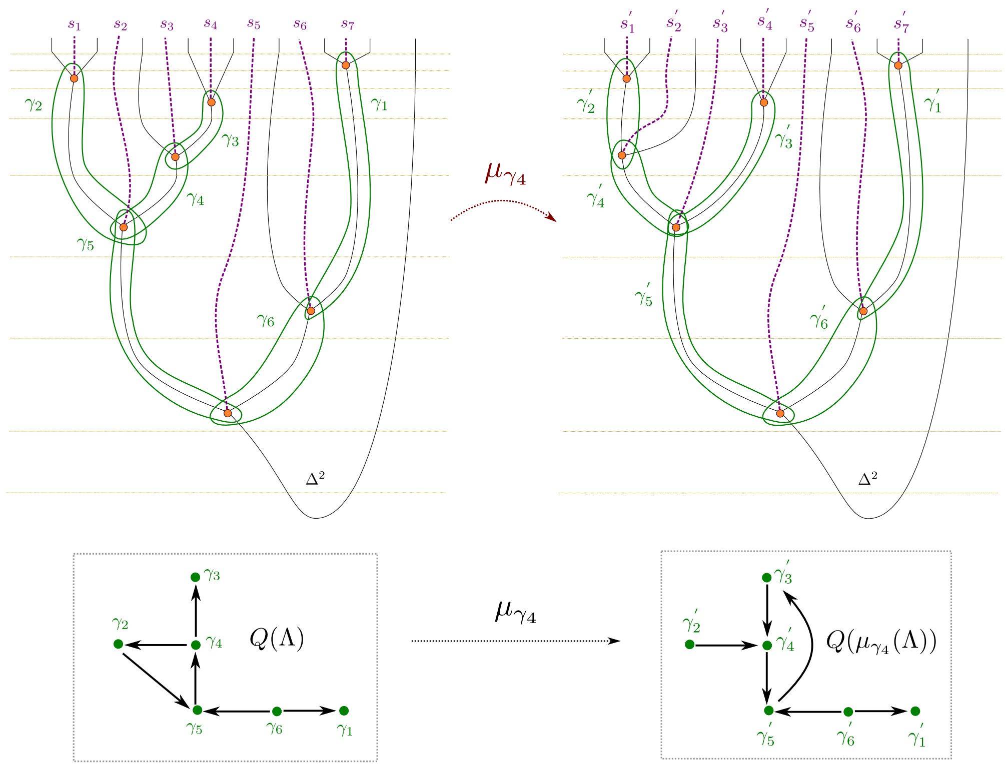


FIGURE 13. A weave mutation between 2-weaves and the associated intersection quivers.

Example 6.6. Consider the Legendrian 2-weave Λ in Example 6.5 above and perform a 2-graph mutation along the short I-cycle γ_4. This yields a new Legendrian 2-weave $\mu_{\gamma_4}(\Lambda)$. These two 2-weaves and the mutation are depicted in Figure 13. Let s'_i, $i \in [1,7]$ and γ'_i, $i \in [1,6]$ be the new (variables associated to) relative and absolute 1-cycles associated to $\mu_{\gamma_4}(\Lambda)$, respectively. We want to compare the cluster charts associated to Λ and $\mu_{\gamma_4}(\Lambda)$ by the above procedure, and show that they indeed differ by exactly one cluster A-mutation. First, the intersection quivers $Q(\Lambda), Q(\mu_{\gamma_4}(\Lambda))$ associated to $\{\gamma_i\}, \{\gamma'_i\}$ are drawn in Figure 13, and they indeed differ by a quiver mutation at the vertex γ_4. Second, we need to compare the cluster A-coordinates. For Λ, the cluster chart is given by

$$C(\Lambda) = \{P_{13}, P_{16}, P_{36}, P_{46}, P_{69}, P_{79}\},$$

as computed in Example 6.5. For $\mu_{\gamma_4}(\Lambda)$, the cluster A–coordinates are computed similarly. Indeed, we first express the absolute 1-cycles $\{\gamma'_i\}$ in terms of the relative 1-cycles $\{s'_i\}$:

$$\gamma'_1 = s'_7 = (s'_1 s'_2 s'_3 s'_4 s'_5 s'_6)^{-1}, \quad \gamma'_2 = s'_1, \quad \gamma'_3 = s'_4,$$
$$\gamma'_4 = s'_1 s'_2, \quad \gamma'_5 = s'_1 s'_2 s'_3 s'_4, \quad \gamma'_6 = s'_6 s'_7 = (s'_1 s'_2 s'_3 s'_4 s'_5)^{-1}.$$

Then we express our z_i-coordinates in terms of the s'_i-variables by noticing that the Lagrangian filling for $\mu_{\gamma_4}(\Lambda)$ corresponds to the pinching sequence $(7,1,4,2,3,6,5)$. To ease notation, we write s_i for s'_i, keeping in mind that these are unrelated to the s_i-variables in Example 6.5. A computation yields

$$z_1 = s_1, \quad z_2 = s_2 - \frac{1}{s_1}, \quad z_3 = s_3 - \frac{1}{s_2} - \frac{1}{s_4}, \quad z_4 = s_4, \quad z_5 = s_5 - \frac{1}{s_4} + \frac{1}{s_3 s_4^2} - \frac{1}{s_6}, \quad z_6 = s_6 - \frac{1}{s_7},$$

$$z_7 = s_7, \quad z_8 = \frac{s_5\left(s_6 - s_6^2 s_7\right) - 1}{s_5 s_6^2 s_7^2}, \quad z_9 = s_5\left(s_2\left(s_1 s_2 - 1\right) s_3^2 s_5 s_4^2 + s_3 s_5 s_4^2 + 1\right) s_6^2 s_7^2.$$

The cluster A-coordinates $A'_i = A(\gamma'_i) = A(\gamma'_i(s'(z))) = A(z)$ then become:

$$A'_1 = z_7, \quad A'_2 = z_1, \quad A'_3 = z_4, \quad A'_4 = z_1 z_2 + 1, \quad A'_5 = z_3 z_4 + z_1\left(z_4 + z_2\left(z_3 z_4 + 1\right)\right) + 1, \quad A'_6 = 1 + z_6 z_7.$$

Except for A'_4, highlighted in blue, these coincide with the cluster A-coordinates for $C(\Lambda)$, $A_i = A'_i$, $i \in [1,6]$, $i \neq 4$. The cluster A-coordinate that has changed is $A_4 = z_3 z_4 + 1$, which has become $A'_4 = z_1 z_2 + 1 \neq A_4$. This concludes that the cluster charts associated to $C(\Lambda)$ and $C(\mu_{\gamma_4}(\Lambda))$ differ precisely by a cluster A-mutation. Finally, in terms of the Plücker coordinates P_{ij}, the cluster A-coordinates for $C(\mu_{\gamma_4}(\Lambda))$ are

$$C(\mu_{\gamma_4}(\Lambda)) = \{P_{13}, P_{16}, P_{14}, P_{46}, P_{69}, P_{79}\},$$

which readily differ from the previous cluster chart $C(\Lambda)$ exactly at the coordinate P_{14}, highlighted in blue. Note that we can consider the cluster mutation of the A-coordinates in $C(\Lambda)$ at the vertex γ_4, which is associated to $P_{36} = z_3 z_4 + 1$. From the quiver $Q(\Lambda)$, we get that all A-coordinates remain the same except for the function $c(\gamma_4) = s_3 s_4 = (z_3 z_4 + 1)|_{C(\Lambda)}$ associated to the vertex γ_4, which changes from $P_{36} = c(\gamma_4)$ to P_{14}. This can also be verified combinatorially: the parallelogram that contains the (36)-diagonal in the triangulation dual to the 2-weave Λ has vertices $\{1,3,4,6\}$, and thus the other diagonal is (14), which yields P_{14}. This example concludes our study of the $n = 2$ case. □

6.2. **The General Case.** The structure for the general case of n-weaves, $n \geq 3$, is still given by the same three steps as in Subsection 6.1. Nevertheless, each of the steps is itself combinatorially more elaborate due to the presence of Y-cycles. In fact, we presently do not know of an analogue of Lemma 6.3 for a general weave, and thus we will assume a basis $\{\gamma_i\}$, $i \in I$, of absolute cycles, represented by (possibly long) I- and Y-cycles is given.[16]. Following the scheme above, we first trace the (relative) homology classes given by the boundaries of holomorphic strips, which yields the s_i-variables, and then use the intersection pairing to find which monomials in the s_i-variables are cluster A-coordinates.

Let Λ be an n-weave and $\{\gamma_i\}$, $i \in I$, a basis I- and Y-cycles. For the first step, we construct a relative cycle s_i associated to each trivalent vertex $v_i \in \Lambda$. The relative cycle s_i is specified by a sequence of edges of the n-weave, specified by the following three rules:

(i) The path s_i starts at the upper left segments of the trivalent vertex v_i and moves upwards along that segment.

[16]This assumption is met in all examples we have studied.

(ii) If the path s_i encounters a trivalent vertex as its moves upwards, the it continues upwards using the *right* upper edge for the trivalent vertex.

(iii) If the path s_i encounters a hexavalent vertex as its moves upwards, it passes through it following the upper segment with the same slope with which the path meets the vertex. That is, if s_i enters the hexavalent vertex from the lower right (left) then it moves upwards exiting from the upper left (right); if s_i enters the hexavalent vertex from the lower center then it moves upwards exiting from the upper center.

Examples of such paths are drawn as dashed purples lines in Figure 14.

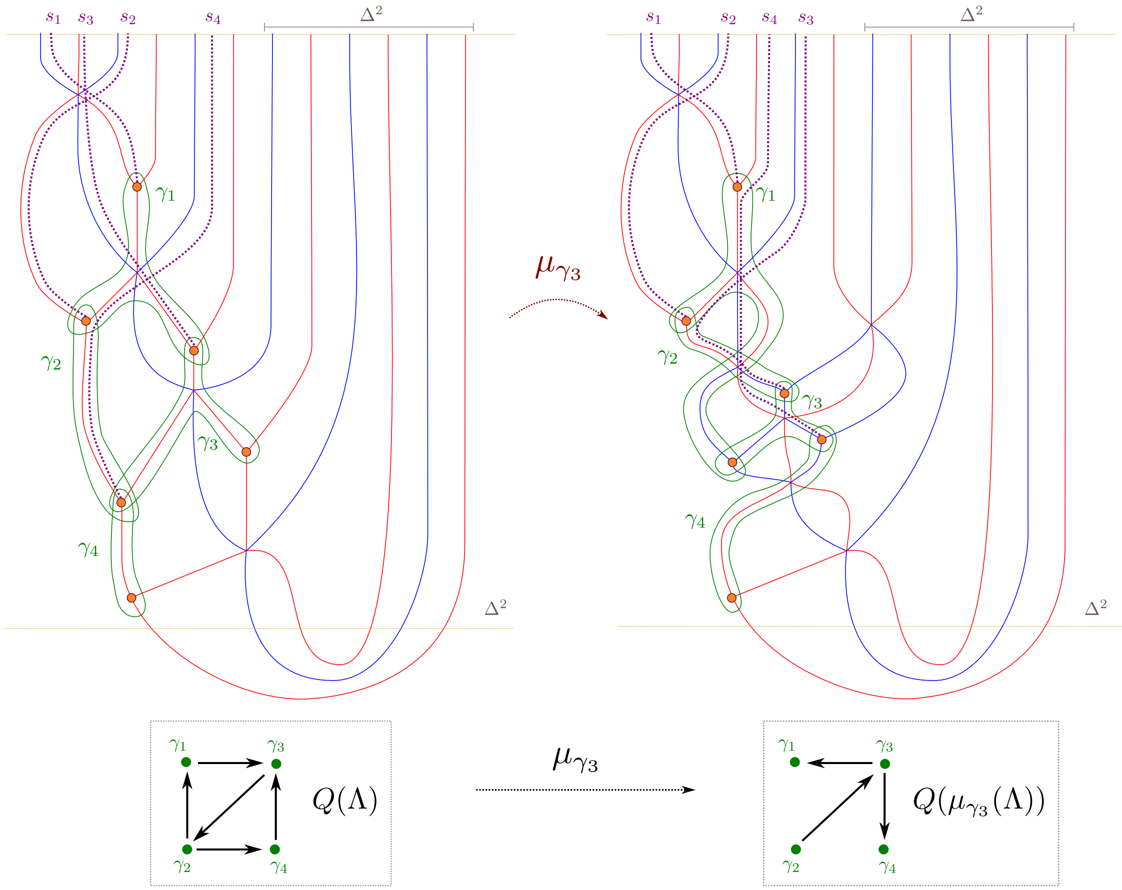


FIGURE 14. Two 3-weaves Λ (Left) and $\mu_3(\Lambda)$ (Right) representing two distinct Lagrangian fillings of the max-tb Legendrian representative of the $(3,3)$-torus link. The cluster A-coordinates are monomials in the s_i-coordinates associated to the trivalent vertices: the monomials are obtained by computing the intersection numbers between the absolute 1-cycles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the relative s_i-cycles represented by dashes lines, which is done in Example 6.7. The corresponding intersection quivers $Q(\Lambda), Q(\mu_{\gamma_3}(\Lambda))$ are also drawn in the bottom of each 3-weave.

Such a path in the n-weave specifies a unique relative homology class by lifting near v_i to the jth and $(j+1)$th sheets [19, Section 2.4], if v_i is a trivalent vertex labeled with the permutation $(j, j+1)$. A simple generalization of Lemma 6.4 shows that there exists a unique holomorphic strip with one positive puncture in the homology class s_i [21, 17]. Finally, for the second step, the intersections between the s_i-relative cycles and the γ_i-absolute cycles are computed according to the rules in [19]. Let us end this section with an explicit example of an algebraic 3-weave and the cluster chart it defines in the corresponding augmentation variety, which is of cluster D_4-type.

Example 6.7. Consider the positive braid $\beta = (\sigma_1\sigma_2)^6$, the link $\Lambda(\beta)$ is the unique max-tb Legendrian representative of the $(3,3)$-torus link. In this case, the augmentation variety is of type D_4 since the brick quiver for $\Lambda(\beta)$ is mutation equivalent to the D_4-quiver. Figure 14 depicts two free 3-weave, Λ (Left) and $\mu_{\gamma_3}(\Lambda)$ (Right), with boundary β, which represent thrice-punctured Lagrangian tori $L(\Lambda), L(\mu_{\gamma_3}(\Lambda)) \subseteq (\mathbb{D}^4, \omega_{\text{st}})$ filling $\Lambda(\beta) \subseteq (\mathbb{S}^3, \xi_{\text{st}})$. Let us first focus on Λ: a basis for $H_1(L(\Lambda), \mathbb{Z})$ is given by the four absolute 1-cycles γ_i, $i \in [1,4]$, where γ_1, γ_3 are short Y-cycles and γ_2, γ_4 are short I-cycles. The relative 1-cycles $s_i \in H_1(L(\Lambda), \Lambda(\beta), \mathbb{Z}))$, $i \in [1,5]$, are shown in dashed purple in Figure 14 (Left). The intersection rules between cycles in a weave [19, Section 2] lead to

$$s_1 = \gamma_1, \quad s_2 = \gamma_1^{-1}\gamma_2, \quad s_3 = \gamma_3, \quad s_4 = \gamma_2^{-1}\gamma_3^{-1}\gamma_4.$$

Hence, the four cluster A-coordinates in the cluster chart induced by $L(\Lambda)$ are given by the monomials

$$A_1 = s_1, \quad A_2 = s_1 s_2, \quad A_3 = s_3, \quad A_4 = s_1 s_2 s_3 s_4.$$

The cluster X-coordinates in this chart can be read by using the intersection quiver $Q(\Lambda)$ of the $\{\gamma_i\}$, shown in Figure 14 (Left), which is mutation equivalent to the D_4-quiver. Second, let us conclude by performing a 3-weave mutation to Λ at the short Y-cycle γ_3. The resulting 3-weave $\mu_{\gamma_3}(\Lambda)$ is depicted in Figure 14 (Right). In this case the intersections read

$$s_1 = \gamma_1, \quad s_2 = \gamma_1^{-1}\gamma_2, \quad s_3 = \gamma_2^{-1}\gamma_3, \quad s_4 = \gamma_3^{-1}\gamma_4.$$

In consequence, the cluster A-coordinates in the cluster chart of the associated Lagrangian filling $L(\mu_{\gamma_3}(\Lambda))$ are

$$A_1 = s_1, \quad A_2 = s_1 s_2, \quad A_3 = s_1 s_2 s_3, \quad A_4 = s_1 s_2 s_3 s_4,$$

which differ from the cluster A-coordinates above exactly at A_3, as desired. □

The articles [16, 19] illustrate that the diagrammatic calculus of algebraic weave is able to combinatorially access *infinitely* many clusters in many cluster algebras. The above instances show that this can be done in an algorithmic and explicit manner, and a complete description of a *cluster* weave calculus will be the object of future work.

7. Future Directions

Finally, we would like to list a few problems and directions for future research. These are currently stated in an open-ended manner, we hope that they might be helpful to the interested reader.

7.1. **Subword Complexes.**

First, the combinatorics of weaves appears to be closely related to the combinatorics of *subword complexes* and brick polytopes [12, 22, 32, 42, 43, 51, 59, 60, 78]. The faces of a subword complex for a braid word γ correspond to all possible subwords of γ such that the Demazure product of their complements equals w_0. Subword complexes were introduced by Knutson and Miller [59, 60] in the context of Gröbner geometry of Schubert polynomials. Knutson and Miller proved that subword complexes are homeomorphic to balls or spheres. Pilaud and Stump found polytopal realizations of spherical subword complexes and called them *brick polytopes.* Escobar [32] related toric varieties of brick polytopes to Bott-Samelson varieties and to Brion's resolutions of Richardson varieties. Results of [42, 43] describe the behavior of subword complexes under braid moves and moves $s_i s_i \to s_i$ in γ. Ceballos, Labbé and Stump [22] proved that certain brick polytopes are generalized associahedra, thus relating subword complexes to the theory of cluster algebras. See also more recent works of Brodsky and Stump [12] and of Jahn, Löwe and Stump [51] further exploring this relation.

In some cases, the collection of all (minimal in an appropriate sense) Demazure weaves between a given braid word and its Demazure product appears to have a polytope-like structure, see Example 4.16. More precisely, the Hasse graphs of principal down sets of some elements in our version of the second Bruhat order seem to admit realizations as the $1-$skeleta of certain polytopes. We conjecture that the corresponding simplicial complexes for arbitrary elements are always homeomorphic to balls or spheres, in analogy to subword complexes. In the spherical case, we conjecture to always have such polytopal realizations. Note that McConville [67] proved that intervals in the second higher Bruhat orders are contractible or homotopy equivalent to spheres.

Nevertheless, the combinatorial structures of weaves and of subword complexes are rather different. For example, for the braid word s_1^n on two strands there are Catalan many non-equivalent Demazure weaves, which correspond to the vertices of the associahedron, while the subword complex is a simplex.

On the other hand, the associahedron also appears as a subword complex for the n-strand braid word $c\Delta$ where c is the Coxeter element. It would be very interesting to find a precise relation between the two theories. It would be also interesting to compare the combinatorics of weaves to cluster structures on Richardson and Schubert varieties constructed in [62, 83]. Note that Leclerc and Zelevinsky discussed the link between second higher Bruhat orders and wiring diagrams (and pseudoline arrangements) for reduced braid words in [63].

7.2. **Open Soergel Calculus.** Second, the relation between our weave category $\mathfrak{W}_n$ and the Soergel calculus developed in [30, 31] is not a coincidence. Indeed, the latter describes the morphisms between the products of Bott-Samelson bimodules B_i in the category of Soergel bimodules. Geometrically, products of Bott-Samelson bimodules correspond to Bott-Samelson varieties, while our braid varieties (or open Bott-Samelson varieties in [86]) naturally correspond to the products of Rouquier complexes $T_i = [B_i \to R]$.

It is therefore natural to expect that the weave calculus corresponds to the desciption of the homotopy category of Soergel bimodules as a monoidal (dg) category generated by the products of T_i. In particular, 6- and 4-valent vertices correspond to homotopy equivalences [82]

$$T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1},\ T_i \otimes T_j \simeq T_j \otimes T_i\ (|i-j|>1),$$

while cups, caps and trivalent vertices correspond to the maps in the skein exact triangle

$$[T_i^2 \to R] \simeq [T_i \to T_i].$$

It is plausible that the equivalence relations from Section 4.2 hold on Soergel side, as they match the corresponding relations for B_i up to lower order terms.

The main obstacle in developing a diagrammatic calculus for T_i is that they generate a dg category (while B_i generate an additive category), and one needs to keep track of homotopies and possible higher A_∞ products on morphisms. Still, it is plausible that some of these higher structures are visible in braid varieties. For example, the variables z_i assigned to crossings (or rather 1-forms dz_i) seem to match the dot-sliding homotopies in Soergel calculus, and it was observed in [41] that the symplectic form on the braid variety corresponds to a certain explicit operator on Rouquier complexes of homological degree (-2). We plan to investigate the relation between the differential forms on braid varieties and Rouquier complexes in the future work.

Finally, both subword complexes and Soergel calculus can be defined for any Coxeter group (with a realization), and it would be also interesting to develop the combinatorics of weaves and braid varieties for braid groups outside of type A. In particular, the 4- and 6-valent vertices should be replaced by an $2m_{ij}$-valent vertex for any Coxeter relation $(s_i s_j)^{m_{ij}} = 1$, and one needs to generalize the change of variables (2.2) to this case. Abstractly, the existence of such change of variables is guaranteed by [84, Section 2.2], but the symplectic and contact geometric meaning of these diagrams remains to be discovered.

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