MAT 108: FINAL PROJECT

DUE TO SUNDAY DECEMBER 13 2020

This contains the instructions for your Final Project in "MAT 108 Introduction to Abstract Mathematics", and a list of suggested topics. **Please read the instructions carefully** before you proceed to choosing the topic. Understanding the precise nature of the project is a helpful step towards an appropriate choice of topic. The Final project must be submitted through Gradescope by Sunday, December 13 at 9:00am.[1]

1. Instructions for your Final Project

In this section we describe the instructions for the final project, including its structure, ingredients that must be part of the project and some general tips.

1.1. Structure of the Project.

You must adhere to the following structural and logistic aspects when writing and submitting your Final Project:

(a) Your Final Project must have a clearly stated central topic. This topic must be a result or problem of *mathematical* nature.

(b) You can choose the central topic of your Final Project. Several suggestion can be found in Section 2 below: you may choose from them or follow your own initiative.

It is possible that two classmates have chosen a common, or intersection, topic. This is *not* a problem as long as the two projects are developed independently, and will only be evaluated on their own individual merits.

(c) The Final Project must be at least *four pages* long in content, not including the title pages, table of contents or references. There is no upper limit.[2]

(d) The Final Project will be submitted *through Gradescope*. There is no specific format required regarding the writing and editing of the project: it can be written by hand, through a word editor, with LaTeX or in any other manner. You will submit a *PDF file* to Gradescope, and only that final file will be evaluated.

(e) If your Final Project uses a computer code or an algorithm that you have typed yourself, do include such code at the end of the project.

[1] There is *no* Final Exam, instead it has been substituted by this Final Project.

[2] Nevertheless, please do read the tips below and aim for a reasonable length.

1.2. **Mandatory Aspects.** The following list specifies items that **must be included** in the Final Project.

1. The **targeted audience for your project must be a generic classmate** for our "MAT-108 Introduction to Abstract Mathematics". That is, it should be written and intended for one of your peers in this Fall 2020 course. You *may not* assume that your peers know anything beyond what we will cover in MAT-108. In particular, **any of your MAT-108 classmates** must be able to read *and* understand each of the parts of your Final Project without having to access any external sources: it must be accessible and self-contained.

2. The project **must contain the proof of at least one theorem**. If the central problem being discussed in your project is a proven Theorem, then you might include a proof of that Theorem, or just a proof in a particular case. For instance, you may add hypothesis that simplify the argument, and provide the proof in the simplified case.[3]

   If you central problem being discussed is a Conjecture, you must include the proof of a simpler version of that conjecture, or a particular case, or a proof solving a certain problem that is closely related to the conjecture. In the latter case, you must provide a clear explanation about the relation between what you are proving and the central topic of your project.

3. The project **must contain all the mathematical preliminaries** clearly explained, with motivation and examples, such that *any* of your peers that have taken our "MAT-108 Introduction to Abstract Mathematics" would be able to follow the proof in the first item if they read your project. In particular, the final project **must contain examples**.

4. The project **must contain all the references** being used, including articles, books, online resources (such as websites, video lectures or blogs) or conversations with somebody that helped. The latter can be included in the *acknowledgements* sections or as a "Private Communication" at the references in the end. References must be included *at the end* of the article in an alphabetically ordered reference list.[4]

Please read each of the above mandatory aspects carefully: it will be easier to choose a topic and develop your project if you have a clear understanding of the most important aspects that must be included.

[3]For example, suppose that your project discusses Fermat's Last Theorem that there are no non-zero integral solutions to $x^n + y^n = z^n$. Despite this being proven as of 2010, you can just include the proof for the case that $n = 3$, or $n = 4$ or n being a prime: whichever case you feel you understand and can be explained.

[4]For an example, see pages 110-114 in this article: https://www.math.ucdavis.edu/ casals/LegendrianWeaves.pdf.

Grade. In evaluating and grading your Final Project, the following items will be of particular focus:

(i) Mathematical correctness.

(ii) Correctly adhering to the **mandatory aspects** instructed above.

(iii) Clarity of the written exposition. At least, this entails writing clear explanations, including illustrative and motivating examples and presenting the material in a self-contained and organized manner.

Any indication that the Code of Academic Conduct of the UC Davis Office of Student Support and Judicial Affairs has been violated will immediately grant a zero grade.[5]

Note that the particular choice of topic will have no weight in the grade. In particular, a well-written exposition on a simple topic might be granted full grade.[6]

In the same vein, unjustified or unexplained use of complicated or advanced terminology does almost always hinder the reader's ability to understand a manuscript, and thus it is strongly advised that you thoroughly refrain from using words and mathematical concepts that do not bring anything to the reader. Please see above for the specified target audience that for your manuscript. In short, we will be much more impressed by **excellent and clear** writing, **self-contained exposition** and an abundant use of illustrative and **motivating examples** than by apparently complicated concepts: the goal is that the reader, in this case one of your classmates, is able to learn and get something out of it.

1.2.1. *Standard Mistakes.*

(i) Choosing a topic that is too hard and inaccessible. To avoid this mistake, choose a topic that you have a chance of learning well: it is important that it is new and exciting, but at the same that you must be realistic towards what you can cover in a *self-contained* project starting with the material taught in "MAT-108 Introduction to Abstract Mathematics".

(ii) Aiming for a *long* Final Project. To avoid this mistake, choose a topic that can actually be explained in a reasonable number of pages. Incredibly deep mathematics fit into short and concise essays: for example, J.F. Nash's thesis was 26 pages (beautiful theorem, and a Nobel prize in Economics), K. Gödel revolutionized mathematics in 1930 with 11 pages[7]. Choose a clear problem *and get to the point* of explaining why it is interesting[8] and what can be said about it.

(iii) Comparing your project to the project of a classmate. To avoid this mistake, focus on what *you* have to say and do not be impressed by the apparent depth or aspects of another project you might see. Talking to your peers is constructive and they can give you good ideas, but *do not* doubt yourself: follow *your*

[5]Please see the Course syllabus for further specifics.

[6]Similarly, a poorly written project on an advanced topic will be granted a poor grade.

[7]His thesis is explained in the short article "Die Vollständigkeit der Axiome des logischen Funktionenkalküls".

[8]Especially, why it is interesting *to you*.

instincts about what *you* think is the right topic for you, the right way that *you* think it should be explained and why *you* find it exciting.

1.2.2. *Additional Tips.*

(i) **Only write what you fully understand**. In mathematics, it is crucial to understand and be able to prove every statement that you claim. In particular, strongly avoid using facts that you do not know about.[9]

(ii) **Give yourself time**. This includes giving yourself time to learn concepts, time to work through examples, time to appreciate the subtleties of a statement, and time to decide how you are going to present things. Then give yourself time to write things.[10]

Start with enough time ahead, which will allow you to have the time to overcome any mistakes or difficulties that might occur in the process. You have an entire month: start early and give yourself the space and time to develop a good project. If in the end the project is short, then it is short: good and short is better than bad and long.

(iii) **Writing is re-writing**. The process of acquiring knowledge is as intricate as that of *communicating* knowledge. Start with a draft, then edit it, then edit again, and then edit again and so on. Take breaks in the meantime, and then try again. At some point you will start to be satisfied with what you have, then you let it rest for a few days and come back at it for a final polish.

2. List of Suggested Projects

We include here a list of topics that might be suitable for a Final Project. As explained above, you are welcome to choose your *own* topic, regardless of whether it is related to any of the suggested projects below. The suggestions might contain a reference or not, and you might, or might not, decide to use a reference if it is provided. Each of the suggestions provides a title: it is part of your work to decide in how much depth you will delve into the topic as well as the exact structure of your own Final Project.

Choose a project that you are excited about. This Final Project is not about impressing anybody, it is just about you enjoying and discovering the mathematics on a subject that sparks your interest, and then learning how to write clearly about it. The grade will solely be based on the quality of the exposition, clarity and self-containment of the presentation and mathematical correctness.[11]

In short, *you* will choose your own topic and learn about it. That includes finding the sources and references that are most helpful to you: they might be articles, books,

[9] If you are using facts that you *do* know about but have not been covered in the MAT-108, then you need to include an explanation about them. Remember that your target audience is a MAT-108 peer.

[10] It would be too ambitious, almost ridiculous, to train for a marathon in a week if you have never run. It is equally non-sensical to believe that one can acquire deep knowledge, digest, process it and communicate it successfully in a week.

[11] In particular, our evaluation and corresponding grade will be *independent* of the specifics of the chosen topic.

online sites or *any* other resource that helps you understand and learn about the topic.

Example: Suppose your topic of choice is "Potatoes". You must then learn about the topic and decide which parts will your project focus on. For instance, you might want to explain the mathematical history and motivation for the "Theorem of the Mashed Potato", providing many examples, a proof for it, or at least a proof in some cases (e.g. assuming the potato is peeled). As another example, you might want to focus on how potatoes are used in applied math - or Computer Science, or Economics, or anything - and explain the mathematical results that make potatoes useful in those fields, again with clear mathematical statements, motivating examples and some proofs. □

Here are some *suggested projects*:

1. *Appendices in "The Art of the Proof"*, which are:
   (a) Continuity and Uniform Continuity,
   (b) Public-Key Cryptography,
   (c) Complex Numbers,
   (d) Groups and Graphs,
   (e) Generating Functions,
   (f) Cardinal Numbers and Ordinal Numbers,
   (g) Remarks on Euclidean Geometry.
2. *Visual arguments*: Proofs with no words.
3. *Planar Graphs and Euler's Formula.* For instance, read Chapter 12 of "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelik'an and K. Vesztergombi. Note that Euler's Formula, and more generally the Euler characteristic, has many applications. Some examples of suggested problems that you can discuss are:
   (a) Show that K_5 is a non-planar graph.
   (b) Show that the only 2-dimensional regular polyhedra are the Platonic solids.
   (c) The Euler characteristic of a triangulation of a closed compact surface is independent of the triangulation.
4. *Infinitely Many Primes.* Some suggested lines of investigation to discuss are:
   (a) Different Proofs of the Existence of Infinitely Many Primes. For instance, read Chapter I of the Number Theory Section in "Proofs from The Book" by M. Aigner, G.M. Ziegler.
   (b) E.g. Infinitely Primes of the form $4k + 1$, and of the form $4k + 3$. Why is the former more difficult to prove ?

(c) E.g. study whether there are infinitely Primes of the form $6k+1$, $6k+3$ and $6k+5$. (You must provide proofs !)
(d) When can Euclid's argument be adapted to show infinitely many primes of the form $ak+b$?
(e) Dirichlet's theorem on arithmetic progressions.

5. *Irrationality of Certain Real Numbers.* Suggested problems are:

   (a) Show that π or e are irrational.
   (b) Show that e^k is irrational for all $k \in \mathbb{N}$. Explore the irrationality of π^k. For e^2 and π^2, read Chapter VI of the Number Theory Section in "Proofs from The Book" by M. Aigner, G.M. Ziegler.
   (c) Irrationality of Liouville Numbers.

6. *Identities Between Fibonacci Numbers.* For instance, read Chapter 4 of "Discrete Mathematics: Elementary and Beyond" by L. Lovász, J. Pelik'an and K. Vesztergombi. One could also study other sequences defined by recurrence, such as the Lucas Sequence, or the Pell Sequence.

7. *Geometry and modular arithmetic.* Explore the geometry using $\mathbb{Z}_p$, i.e. *geometry modulo p*. Some of the first questions you may want to address are:

   (a) What does it mean to do geometry with modulo p ?
   (b) How do we do linear algebra modulo p ?
   (c) How many lines are there in the plane $\mathbb{Z}_p^2$?
   (d) How many lines and planes are there in the space $\mathbb{Z}_p^3$?
   (e) Explore the relation to q-numbers and q-combinatorics.

8. *The Collatz Conjecture*: There are many notes out there, see for instance "The Collatz conjecture aka the $3n+1$ problem" by M. Weitzer, or start at the Wikipedia Entry on the Collatz Conjecture.

   (a) Discuss history of the problem and some attempted strategies.
   (b) Similar problems and generalizations.
   (c) The recent progress by T. Tao in this conjecture. See his blog and his article.

9. *The Catalan Recursion*: You can start by exploring where Catalan numbers appear (in *so* many places) by reading Catalan Numbers by R. Stanley. For instance,

   (a) Find five different objects counted by Catalan numbers, and prove this.
   (b) Construct explicit bijections between some of these objects.
   (c) Prove some of the recursions and formulas for the Catalan numbers C_n.

10. *Modular Arithmetic and Cryptography.* How to use *modulo n* to find good encryption protocols. For instance, you may discuss the factorization problem, or the notion of *discrete* logarithms modulo n, as a few of the basic mathematical

problems in cryptography.

11. *The Binomial Theorem*: You may discuss different proofs of this theorem, or applications. It might also be interesting to generalize the theorem for expansions of $(x+y)^\alpha$ when $\alpha \in \mathbb{R}$ is a real number, or generalize to the *multinomial theorem* or the connection to anagrams.

12. *The Gamma Function*: Generalizing the factorial $n!$ to all $\mathbb{R}$. You may want to understand the definition of the Gamma Function, its motivation. Then prove some of its properties, and discuss how it relates to the material in MAT-108, especially the $n!$ numbers.

13. *The construction of Real Numbers* $\mathbb{R}$: You can start with the Dedekind cut, or survey the history of how mathematicians rigorously defined the real numbers, for example. Some instances of techniques or topics related to the construction of Real Numbers $\mathbb{R}$ are:

    (a) Using Dedekind cuts,
    (b) Via completion of another field (the rational numbers $\mathbb{Q}$),
    (c) Discuss or prove uniqueness of the complete ordered field,
    (d) Construct other completions, like p-adic numbers.[12]

14. *Algebraic Numbers*: Explore the definition of these irrational real numbers which are, in a sense, *a bit* rational. Prove some of their properties and give many examples, always justified and with explanations. Some examples that are illustrative to explore and discuss are:

    (a) Gaussian Integers.
    (b) Algebraic Integers.
    (c) Examples of non-algebraic irrationals.

    Similarly, one may alternatively explore *transcendental numbers*, such as e and π, which are those irrational number that, in a sense, are very much not rational.

15. *Cardinality*: One may explore different notions of cardinality, with definitions and examples. Which set has *more elements* $\mathbb{Z}$ or $\mathbb{Q}$? And between $\mathbb{Q}$ and $\mathbb{R}$? These are really interesting questions, and you can already start reading about some interesting concepts here:

    (a) Continuum hypothesis.
    (b) Aleph Numbers.

16. *Euler's totient function and Euler's Theorem*: This is a generalization of Fermat's Little Theorem for non-prime numbers. Thus this would be a project related to modular arithmetic, now doing *modulo* n where n is not necessarily

[12]Or, if you want, other non-Archimedean fields.

a prime, and trying to understand what a^n is equal to *modulo* n.

17. *Polynomial Interchanges and Separable Permutations.* For instance, read Pages 21-25 from É. Ghys "A Singular Mathematical Promenade".

18. *Different Number Systems*: Explore number systems beyond $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$, maybe introducing and discussing the complex number $\mathbb{C}$. You may also, or alternatively, discuss more exotic number systems, such as:

    (a) Quaternion Numbers,
    (b) Octonion Numbers,
    (c) Finite Fields.

19. *Fixed Point Theorems* in Real Analysis, and also their applications. For instance, the "Kakutani fixed-point theorem" and its application to the existence of Nash equilibra in non-cooperative games.

20. *Recurrences* in Computer Science and Dynamics Programming. You might also want to explore the *Master Theorem*[13] in analysis of algorithms.

21. *Singmaster's Conjecture*: This question was first asked in Singmaster's paper: How often does an integer occur as a binomial coefficient? You can also start at the Wikipedia Entry on Singmaster's conjecture. Here are some questions to consider:

    (a) We know the number 1 occurs infinitely many times. Can other numbers occur infinitely many times?
    (b) Is there a number besides 1 occurring twice, three times, and so on?
    (c) How do you tell if elements of Pascal's triangle are divisible by a prime p? Are infinitely many of them divisible by p?

We end this list of suggested project by emphasizing that you are *free* to choose any project that you find interesting, whether it is in the above list or not. You will be the author of your Final Project, and it is part of your work to decide what is being discussed, in what manner, with which examples and proofs to include. At the end of the day, *only you* can really know what project is most appropriate for your, depending on your mathematical taste, background and time disposition. You may discuss your choices with your peers, friends, or anybody that may be of help, but it is part of the mathematical learning process to work on that ourselves as well.

[13]This gives an asymptotic analysis for algorithms with divide-and-conquer recurrences.