Lecture 14

Tuesday, September 29, 2020 12:05 PM

Lecture 14: The Real Numbers $\mathbb{R}$ (Chapter VIII)

intuitively $\underset{\mathbb{N}}{1}, -3, \underset{\mathbb{N}}{27} \in \mathbb{Z}$

$1/2$? $\sqrt{2}$? e, π? $3.15923\ldots$?

We will consider a new set, denoted $\mathbb{R}$, with operations $(+, \cdot)$ satisfying Axioms 8.1-8.4 (see textbook: same as $\mathbb{Z}$). The new axiom for product:

$\boxed{\text{Axiom 8.5: } \forall x \in \mathbb{R} \setminus \{0\},\ \exists y \in \mathbb{R} \text{ s.t. } x \cdot y = 1.}$ ← y is denoted x^{-1}, or $1/x$, informally "we can use fractions"

↳ not true for $\mathbb{Z}$: if $3 \cdot y = 1$, then $y \notin \mathbb{Z}$.

Similar to how we introduced $\mathbb{N} \subseteq \mathbb{Z}$, $\exists\, \mathbb{R}_{>0} \subseteq \mathbb{R}$ a subset, called "positive real numbers",

Axioms for $\mathbb{R}_{>0}$:
- (i) $x, y \in \mathbb{R}_{>0}$, then $x + y \in \mathbb{R}_{>0}$
- (ii) $x, y \in \mathbb{R}_{>0}$, then $x \cdot y \in \mathbb{R}_{>0}$
- (iii) $0 \notin \mathbb{R}_{>0}$
- (iv) $\forall x \in \mathbb{R}$, either $x \in \mathbb{R}_{>0}$, or $x = 0$ or $(-x) \in \mathbb{R}_{>0}$. (only 1 happens)

Recall that $\mathbb{N} \subseteq \mathbb{Z}$ had a smallest element. ⟶ key to have base case in induction

$\boxed{\text{Thm: the set } \mathbb{R}_{>0} \text{ has no smallest element.}}$ ⟶ induction over $\mathbb{R}_{>0}$ is not possible

Proof of thm: By contradiction, suppose $s \in \mathbb{R}_{>0}$ is the smallest element.

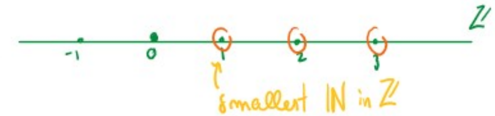
Consider the real number $\frac{s}{2} \in \mathbb{R}$, ↳ by Ax. 8.5

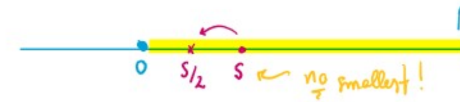
in fact, $\frac{s}{2} \in \mathbb{R}_{>0}$ because

$2 \in \mathbb{R}_{>0}$, so $\frac{1}{2} \in \mathbb{R}_{>0}$ and thus

$\frac{s}{2} = s \cdot \frac{1}{2} \in \mathbb{R}_{>0}$. But $s/2 < s$,

so $s \in \mathbb{R}_{>0}$ is not the smallest element. $\Longrightarrow$ $\nexists$ smallest element! ∎





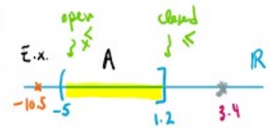
§2. Bounds for subsets

⇝ crucial to talk about limits and other idiosyncrasies of $\mathbb{R}$

Def^n: A subset $A \subseteq \mathbb{R}$ is said to be bounded above by $b \in \mathbb{R}$ if $\forall a \in A$ we have $a \leq b$. ← called "upper bound"

A subset $A \subseteq \mathbb{R}$ is said to be bounded below by $c \in \mathbb{R}$ if $\forall a \in A$ we have $a \geq c$. ← called lower bound

E.x.



$A = \{x \in \mathbb{R} : -5 \leq x \leq 1.2\}$

i.e. $-5 < x \leq 1.2$

(1) upper bounds? Any $x \geq 1.2$

(2) lower bounds? Any $x \leq -5$.

As you see, $\exists$ many upper (and lower) bounds:

Def^n: Let $A \subseteq \mathbb{R}$ be a subset. The least upper bound is called the supremum.

The greatest lower bound is called the infimum.

↳ they might not exist, if they do, then they're UNIQUE!

⇝ leads to the "COMPLETENESS" AXIOM!