Lecture 15

Tuesday, September 29, 2020 12:05 PM

Lecture 15 : Bounds, supremums & infimums

: remember upper bounds, lower bounds then sup & inf.

First, a subset $X \subseteq \mathbb{R}$ might or might not have upper bound

(1) The subset $\mathbb{R}_{>0}$ does not have an upper bound. ($\nexists$ supremum) (but 0, -1 or -3.145 are lower bound)

(2) $X = [3, 7) := \{ x \in \mathbb{R} : 3 \leq x < 7 \}$. Here 7 (or $13, 10^{10}$) is upper bound.

In general, all possible combinations (bounded above / below or not) do occur.

Axiom 8.52 : Every non-empty $X \subseteq \mathbb{R}$ bounded above has a supremum.

i.e. if $\exists$ upper bound, then $\exists$ least upper bound. ← "COMPLETENESS Axiom"

Exercise : (i) If lower bound exists, then $\exists$ infimum. (ii) If infimum or supremum exist then they are unique.

Proposition : Consider $X \subseteq \mathbb{R}$ an interval bounded and below ($\Rightarrow \exists \sup(X), \inf(X)$)

Then

$\sup([a,b]) = b$ $\quad \inf([a,b]) = a$

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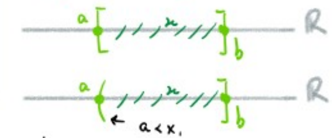
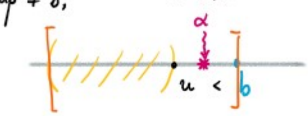
← def^n: $[a,b] \subseteq \mathbb{R}$ is defined as $\{ a \leq x \leq b \}$.



Proof : let's show $\sup([a,b]) = b$. By contradiction, suppose $\sup \neq b$, i.e. $u \in \mathbb{R}$ which is $\sup([a,b]) = u$ with $u < b$ ⊛

Then consider $\alpha = \frac{u+b}{2}$, then $\alpha \in [a,b]$, but

$u = \frac{u}{2} + \frac{u}{2} \overset{⊛}{<} \frac{u}{2} + \frac{b}{2} = \frac{u+b}{2} = \alpha$, so u is not an upper bound. ∎

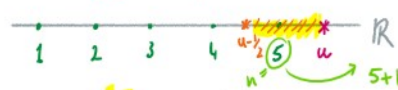


§ 2. The integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$

Inside a $\mathbb{R}$, we have $\boxed{1 \in \mathbb{R}}$, by the axioms : use that to define $\mathbb{N} \subseteq \mathbb{Z}$. (sums of 1) (sums of 1 and −1)

Theorem 10.1. : The subset $\mathbb{N} \subseteq \mathbb{R}$ is not bounded above. — note $\mathbb{N}$ is bounded below, $\inf(\mathbb{N}) = 1$.

Proof : By contradiction, we assume $\mathbb{N}$ is bounded above. By the completeness axiom, $\exists \sup(\mathbb{N})$, call it $u \in \mathbb{R}$. Consider the interval $(u - \frac{1}{2}, u] \subseteq \mathbb{R}$. It must contain a natural number n : indeed, if it did not, then $u - \frac{1}{2}$ would be an upper bound with $u - \frac{1}{2} < u$, so u would not be a supremum. Then $u - \frac{1}{2} < n$, so $u + \frac{1}{2} < n+1$. Since, $u < u + \frac{1}{2} < n+1$ ($\in \mathbb{N}$), then u is not upper bound for $\mathbb{N}$. ∎



Corollary : (i) The integers $\mathbb{Z}$ are not bounded above. (since $\mathbb{N} \subseteq \mathbb{Z}$ & thm. 10.1)

(ii) $\mathbb{Z}$ not bounded below.

Proposition 10.4. : For each $\varepsilon > 0$ (very small number, 0.000000000000001), then

$\exists n \in \mathbb{N}$ such that $\boxed{\frac{1}{n} < \varepsilon}$.

Proof of Prop. : Since $\frac{1}{\varepsilon} \in \mathbb{R}^+$ and $\mathbb{N}$ is not bounded above, then $\exists n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n$; this is equivalent to $\boxed{\frac{1}{n} < \varepsilon}$. ∎

(if $\nexists n$ w/ $\frac{1}{\varepsilon} < n$, then $\frac{1}{\varepsilon}$ would be upper bound for $\mathbb{N}$)

← this will be used all time for limits

Starting on Friday !