Lecture 17

Tuesday, September 29, 2020 12:05 PM

Lecture 17 : Monotone Convergence Theorem

(x_k) seq. of $\mathbb{R}$ numbers.

Prop. 10.14. (Uniqueness of limits) Let $(x_k)_{k\in\mathbb{N}}$ be a convergent sequence, so $\exists L$ s.t. $\lim_{k\to\infty} x_k = L$.
Then L is unique, i.e. if $L_1 = \lim_{k\to\infty} x_k$ and $L_2 = \lim_{k\to\infty} x_k$ then $L_1 = L_2$.

Sol^n: First, if $\varepsilon > 0$ we have that $|L_1 - L_2| < 2\varepsilon$ then $L_1 = L_2$.

$\underbrace{L_2 \quad L_1}_{\varepsilon}$ "$\varepsilon \to 0$"

Let $\varepsilon > 0$ be given, then $\lim_{k\to\infty} x_k = L_1$ means $\exists n_0$ s.t. $|x_n - L_1| < \varepsilon$ for $n \geq n_0$

also $\lim_{k\to\infty} x_k = L_2$ means $\exists n_0'$ s.t. $|x_n - L_2| < \varepsilon$ for $n \geq n_0'$.

$(*)$ $\boxed{\begin{array}{l} |x_n - L_1| < \varepsilon \\ |x_n - L_2| < \varepsilon \end{array}}$ for $n \geq \max(n_0, n_0')$ — large enough

adding

add $0 = x_n - x_n$; $|a \pm b| \leq |a| + |b|$

Now $|L_1 - L_2| = |(x_n - L_2) - (x_n - L_1)| \leq |x_n - L_2| + |x_n - L_1| \leq \varepsilon + \varepsilon = 2\varepsilon$.

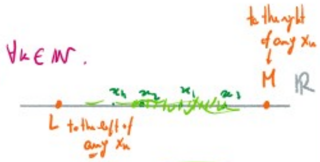
$\left(\text{or} \begin{array}{l} < \varepsilon/2 \\ < \varepsilon/2 \end{array} \right)$

Hence $\forall \varepsilon > 0$, $|L_1 - L_2| < 2\cdot\varepsilon$ thus $\boxed{L_1 = L_2}$. ∎

§ 1. Three definitions : (x_k) a seq. of $\mathbb{R}$ numbers

Def: (i) A sequence is $\boxed{\text{bounded above}}$ if $\exists M \in \mathbb{R}$ s.t. $x_k \leq M$, $\forall k \in \mathbb{N}$.

Similarly (x_k) bounded below if $\exists L \in \mathbb{R}$ s.t. $L \leq x_k$, $\forall k \in \mathbb{N}$



(ii) A sequence is INCREASING if $x_k \leq x_{k+1}$ for $\forall k \in \mathbb{N}$.
Similarly, (x_k) is DECREASING if $x_k \geq x_{k+1}$ $\forall k \in \mathbb{N}$.

⤳ also known as monotone.

"LEGEND"
BA : bound. above
BB : below,
I : increasing
D : decreasing

Examples:
- $x_k = (3/k)^k$: BB, BA, D
- $x_k = k^2$: BB, I (not BA, not D)
- $x_k = 2^k$: BB, I
- $x_k = \frac{-k(k+1)}{2}$: BA, D

- $x_k = (1 + 1/k)^k$: BB, BA, I
- $x_k = (-1)^k$: BB, BA ⤳ not I nor D!
- $x_k = (-1)^k \cdot \frac{1}{k}$: BB, BA, not I nor D!
- $x_k = 2\cdot k!$: BB, I

⤳ a priori no rel^n between (x_k) CONVERGING and increasing or decreasing

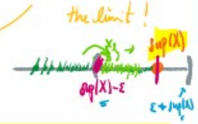
Thm. (Monotone Convergence thm.) Let (x_k) a sequence of $\mathbb{R}$ numbers.

(1) If (x_k) is $\boxed{\text{increasing and bounded above}}$, then (x_k) converges.

(2) If (x_k) is $\boxed{\text{decreasing and bounded below}}$, then (x_k) converges.

change x_k to $-x_k$

we do not know the limit!



Example: $x_k = (1 + 1/k)^k$, it is bounded above (Prob 5 PSet 4) and increasing ⇒ converges and has a limit.

Proof: Let's prove (1). Consider the set $X = \{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Since x_k bounded above, we have X is a set bounded above. By Completeness axiom $\exists \sup(X) \in \mathbb{R}$. We claim $\boxed{\lim x_k = \sup(X)}$

To prove that: given $\varepsilon > 0$, $\sup(X) - \varepsilon$ is NOT an upper bound for X so $\exists$ element $x_\ell \in X$ s.t. $\sup(X) - \varepsilon \leq x_\ell \leq \sup(X)$.

i.e. $x_n \in (\sup - \varepsilon, \sup + \varepsilon)$

In fact, $\forall n \geq \ell$ we have : $\sup(X) - \varepsilon \leq x_\ell \leq x_n \leq \sup(X) + \varepsilon \Rightarrow |x_n - \sup(X)| < \varepsilon$. ∎

$x_\ell \leq x_n$ since x_k increasing; $n \geq \ell$

$\forall n \geq \ell$. $(\ell = n_0)$