Lecture 18

Tuesday, September 29, 2020 12:05 PM

Lecture 18 : Recursive limits & Square Root

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers defined recursively : $\boxed{x_n = \text{formula with } x_1, x_2, \dots, x_{n-1}}$

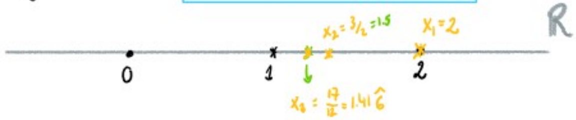
Question : How to show if (x_n) converges ? If so, what is $\lim_{n\to\infty} x_n$?

use Monotone Conv. Thm. (incr. + b.above OR decreas. + bounded below)

how to even guess L ? use recurrence AND UNIQUENESS of limits : $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1}$

given eq^n for $L = \lim_{n\to\infty} x_n$

Today's running example is : $\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \quad x_1 = 2}$



$x_1 = 2$

$x_2 = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2}$

$x_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2\cdot 2}{3}\right) = \frac{1}{2}\left(\frac{9+8}{6}\right) = \frac{17}{12}$

$\boxed{\text{First, show } (x_n) \text{ convergent}}$: we use Monotone Convergence Thm., which we can do if we first prove x_n is bounded below and decreasing :

(i) Bounded below : since $x_1 = 2 > 0$ and sum of positive numbers, $x_n > 0$. → implies bounded below

In fact, $x_{n+1} \geq \sqrt{2}$ because $\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)}$ (#) and so

(*) $x_{n+1} \geq \sqrt{2} \iff \frac{x_n^2 + 2}{2x_n} \geq \sqrt{2} \iff (x_n - \sqrt{2})^2 \geq 0$ true!

true. ⇐ true ⇐ true

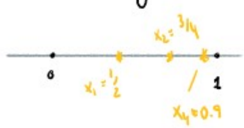
(ii) Decreasing : by induction, and use $x_{n+1} - x_n = \frac{2 - x_n^2}{2x_n} \leq 0$ show this (this follows from (*)).

x_n MCT converg. ⇐ ⇓ $\exists \lim_{n\to\infty} x_n$

$\boxed{\text{Second, compute } \lim_{n\to\infty} x_n}$: using the recursion (#) and uniqueness of the limit $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1}$

Substituting in (#) : $L = \frac{1}{2}\left(L + \frac{2}{L}\right) \Rightarrow$ 2 sol^n L_+, L_-, $L_+ = \sqrt{2} > 0$, $L_- < 0$ $\Rightarrow$ ($x_n > 0 \Rightarrow L > 0$) $L = L_+ = \sqrt{2}$ ▨

Exercise : study the recursive sequence $\boxed{x_{n+1} = x_n \cdot (2 - x_n)}$, $x_1 = 1/2$.



$x_2 = x_1 \cdot (2 - x_1) = \frac{1}{2}\cdot\left(\frac{3}{2}\right) = \frac{3}{4}$

$x_3 = \frac{3}{4}\left(2 - \frac{3}{4}\right) = \frac{3}{4}\left(\frac{5}{4}\right) = \frac{15}{16} = 0.9375$

It seems bounded above and increasing. Let's do bounded above : try $x_n \leq 1$?

$x_{n+1} \overset{?}{\leq} 1 \iff x_n(2 - x_n) \leq 1 \iff 2x_n - x_n^2 - 1 \leq 0 \iff 0 \leq \underbrace{(x_n - 1)^2}_{x_n^2 + 1 - 2x_n}$ true!

try using $x_n \leq 1$

So we have shown $x_{n+1} \leq 1$.

$\boxed{\text{Please do at home : } x_n \text{ increasing AND COMPUTE } \lim_{n\to\infty} x_n \text{ (my bet is } x_n \xrightarrow{n\to\infty} 1 \text{, so } L = 1)}$ ▨

§2. Square roots : how do we define $\sqrt{2}$? (any $\sqrt{r}$ goes the same for $r > 0, r \in \mathbb{R}$)

Consider the set $X = \{x \in \mathbb{R} : x^2 < 2\}$. First, X is bounded above $\Rightarrow \exists \sup(X) \in \mathbb{R}$.

Def^n: We define $\sqrt{2} := \sup\{x \in \mathbb{R} : x^2 < 2\}$. (This is neat, but is this $\sqrt{2}$ really "our $\sqrt{2}$" ?)

Theorem 10.25 : X is bounded above (e.g. 2). ($x < x^2 < 2$ so $x < 2$, so 2 upp. b.) $x \in X$

Also, $\sup(X)$ is such that :

$\boxed{(1)\ \sup(X) > 0 \qquad (2)\ \sup(X)^2 = 2}$

Check Proof in the textbook !

these are the properties that we know and use about $\sqrt{2}$!

⇝ defines rigorously $\sqrt{2}$, and in general same method gets $\sqrt[n]{r}$, $r > 0$, $n \in \mathbb{N}$