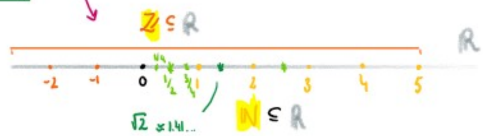
Lecture 20

Tuesday, September 29, 2020 12:05 PM

Lecture 20 : The RATIONAL Numbers

List of some $\mathbb{R}$: $\bullet\ n \in \mathbb{Z}$, Friday

- $\sup\{x \in \mathbb{R} : x^2 \leq 3\} =: \sqrt{3}$ — bounded, so $\exists$ sup
- sol^n to eq^n : $x^3 - 7 = 0 \to$ sol^n $\sqrt[3]{7}$:= sup of set
- Say (x_n) is a convergent seq. : $\lim_{n\to\infty} x_n \in \mathbb{R}$, e.g. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =: e$ Thm $\sum_{n\geq 0} \frac{1}{n!}$



Def^n: A real number $r \in \mathbb{R}$ is RATIONAL if $\exists\, n, m \in \mathbb{Z}$ s.t. $\boxed{r = \frac{n}{m}}$.
A real number $r \in \mathbb{R}$ is IRRATIONAL if it is not rational.

e.g. any $n \in \mathbb{Z}$ is rational ($m = 1$)

↳ basic question : if $x, y \in \mathbb{R}$, $\exists r \in \mathbb{Q}$? $\exists r \in \mathbb{Q}$? x y $\mathbb{R}$

Thm : (Density of $\mathbb{Q}$ in $\mathbb{R}$) Let $x, y \in \mathbb{R}$, $x < y$, then $\exists r \in \mathbb{Q}$ s.t.

$$x < r < y$$

$n, m \in \mathbb{Z}$; n/m ; rational number ; $r = \frac{n}{m}$; x $r \in \mathbb{Q}$ y

if $x, y \in \mathbb{Q}$, then we can take $\frac{x+y}{2} \in \mathbb{Q}$

Proof : By Prop. 10.4, $\exists m \in \mathbb{N}$ s.t. $\frac{1}{m} < \varepsilon$.

$\varepsilon > 0$; x ; $x + \frac{1}{m}$; $\mathbb{Q}$? ; $y = x + \varepsilon$

In order to have $\frac{mx+1}{m}$ with numerator rational, in practice $mx + 1 < n$

since $\mathbb{N}$ unbounded, $\exists n \in \mathbb{N}$ s.t. $\boxed{mx \leq n}$ (Thm 10.1) not nec. $\mathbb{Q}$; $x < \frac{mx+1}{m} < y$; $x \notin \mathbb{Q}$

Hence $\frac{mx+1}{m} \leq \frac{n}{m}$ ← rational! This number might not be less than y. but if we choose

(#) $n \in \mathbb{N}$ to be minimal, then

$$\boxed{x \leq x + \frac{1}{m} \leq \frac{n+1}{m}} \quad \text{and also} \quad \boxed{x < \frac{n}{m} \leq x + \varepsilon = y.}$$

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§ 2. Irrationals exist : we know $\mathbb{Z} \subseteq \mathbb{Q}$, so $\mathbb{Q}$ is non-empty.

check proof in Prop. 11.6 in book!

Question : $\mathbb{Q} = \mathbb{R}$? Equiv. is $\mathbb{R} \setminus \mathbb{Q}$ non-empty?

Thm: $\sqrt{2}$ is irrational, i.e. $\nexists\, n, m \in \mathbb{Z}$ s.t. $\sqrt{2} = \frac{n}{m}$

$\nexists\, d \in \mathbb{Z}$ s.t. $d \mid n$, $d \mid m$

Proof: By contradiction, suppose $\exists\, n, m \in \mathbb{Z}$ s.t. $\sqrt{2} = n/m$, also assume $\gcd(n, m) = 1$.

Since $\sqrt{2} = n/m$, we have $\boxed{m \cdot \sqrt{2} = n}$, which squares to $\boxed{m^2 \, 2 = n^2}$.

CONTRADICTS

From here there's plethora of contradictions : choose one! Let's study divisibility by 2.

(i) Since $2 \mid n^2$, so $2 \mid n$. $\longrightarrow$ $2 \mid n$ and $2 \mid m$ $\Longrightarrow$ $\gcd \geq 2$

(ii) Since $2 \mid n$, then $2^2 \mid n^2$. Hence $4 \mid m^2 \cdot 2$. Now, this implies $2 \mid m^2$, so $2 \mid m$