Lecture 20

Tuesday, September 29, 2020 12:05 PM

Lecture 20: Rational & Irrational Numbers

: $2, \frac{3}{7}, \frac{11}{19}, \sqrt{2}, \sqrt{2}+\sqrt{3}, \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \dots$

rational, i.e. $\mathbb{Q}$ — irrational ($\notin \mathbb{Q}$) — ??? — ?

all reals, i.e. $\mathbb{R}$

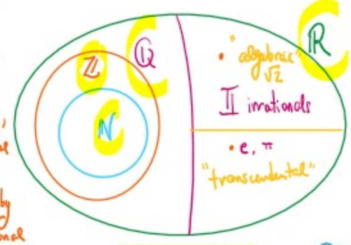
Recall (*): on Monday we proved that $\forall x,y \in \mathbb{R}$, $x<y$ $\exists r \in \mathbb{Q}$ s.t. $x<r<y$. → ∞ many in fact

Theorem: Given any $L \in \mathbb{R}$, $\exists$ sequence (x_n) $(n \in \mathbb{N})$ such that $x_n \in \mathbb{Q}$ and $\lim_{n\to\infty} x_n = L$. → morally, any real can be approx. by rational

Proof: For any $\varepsilon > 0$, including $\varepsilon = \frac{1}{n}$, $\overset{(*)}{\exists}\, r_n \in \mathbb{Q}$ between L and $L+\varepsilon = L+\frac{1}{n}$. Then $x_n := r_n$ converges to L. ∎

Exercise: Show this! — this seq. is not unique!

$L \quad L+\frac{1}{N} \quad L+\frac{1}{n}$ (r_N, r_n)



Cartoon picture

$\sqrt{-1}$ ∉ $\mathbb{R}$

§2. Instances of irrational: from Monday, $\sqrt{2} \notin \mathbb{Q}$ is irrational.

A similar argument shows $\sqrt{6}$, e.g., is irrational. This implies $\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$

↳ $(\sqrt{2}+\sqrt{3})^2 = 2+3+2\sqrt{6} \notin \mathbb{Q} \Rightarrow \sqrt{2}+\sqrt{3} \notin \mathbb{Q}$.

($\underset{\mathbb{Q}}{5} + 2\sqrt{6} \in \mathbb{Q}$? $\Rightarrow \notin \mathbb{Q}$)

A more advanced discussion is about LIMITS, today's example will be $\boxed{x_n = \left(1+\frac{1}{n}\right)^n}$, by PS#4 (Solved) it converges.

Def^n: The limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is called $e \in \mathbb{R}$. ∎ → e is approx. 2.71828…

Question: $\boxed{e \in \mathbb{Q} \text{ or } e \notin \mathbb{Q}\,?}$

(*) follows from Binomial Thm. to $\left(1+\frac{1}{n}\right)^n$; you get: $\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k} n^{-k} = \sum_{k=0}^{n}\frac{1}{k!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}_{\to 1 \text{ as } n\to\infty}$

Lemma: $e \overset{(*)}{=} \sum_{n\ge 0}\frac{1}{n!} \left(= \underbrace{\underbrace{1+1}_{2}+\frac{1}{2!}}_{2.5}+\frac{1}{3!}+\frac{1}{4!}+\dots\right)$.

Thm: e is irrational.

Proof: By contradiction, we assume $\exists p,q \in \mathbb{N}$ s.t. $e = \frac{p}{q}$. By lemma $e = \sum_{k=0}^{\infty}\frac{1}{k!}$.

First: $(*)$ $\boxed{0 < e - \underbrace{\sum_{k=0}^{n}\frac{1}{k!}}_{\text{rational}} \le \frac{1}{n!}}$ ($\notin \mathbb{Q}$): check $\sum_{k=0}^{\infty}\frac{1}{k!} - \sum_{k=0}^{n}\frac{1}{k!} = \sum_{k=n+1}^{\infty}\frac{1}{k!} \underset{\uparrow}{\le} \frac{1}{n!}\Big(\underbrace{\sum_{n\ge 1}\frac{1}{2^n}}_{=1}\Big)$.

bound by geometric series

Second: Since we assume $e = \frac{p}{q}$, we have (#) $\boxed{\frac{p}{q} - \sum_{k=0}^{q}\frac{1}{k!} \overset{(*)}{\le} \frac{1}{q!}}$

The LHS of (#) is:

$$\frac{p}{q} - \sum_{k=0}^{q}\frac{1}{k!} = \frac{p}{q} - \left(1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{q!}\right) = \frac{C}{q!} \quad (C \in \mathbb{Z})$$

clean denom.

$\frac{C}{q!} \le \frac{1}{q!} \Rightarrow C \le 0$

$e - \sum_{k=0}^{q}\frac{1}{k!} = \frac{C}{q!} \le 0 \Rightarrow e \le \sum_{k=0}^{q}\frac{1}{k!}$ → a contradiction! ∎