Lecture 21

Tuesday, September 29, 2020 12:05 PM

Lecture 21 : Introduction to Cardinality ← bijections !

In math there are many "∞"

What has more elements : $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$? ← today we begin to learn how to count "∞"

! : mind-blow !

Thm : $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q})$

AND $\text{card}(\mathbb{Q}) < \text{card}(\mathbb{R})$.

→ (1) Define $\text{card}(X)$, at least if X, Y sets, $\text{card}(X) \lneq \text{card}(Y)$?

→ (2) Compute $\text{card}(\mathbb{N}), \text{card}(\mathbb{Z}), \text{card}(\mathbb{I}), \text{card}(\mathbb{R}), \text{card}(\mathbb{Q})$

Def^n: Let X, Y be two sets, a function $f: X \longrightarrow Y$ is an assignment of one element $f(x) \in Y$ for each $x \in X$.

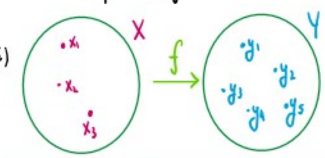
↳ it's fine to have $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$

↳ it's fine if $\exists y$ s.t. $y \neq f(x)\ \forall x \in X$

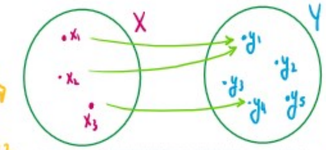
↳ it's not allowed to assign more than one y to an x.

§1. Examples of functions :

(i)

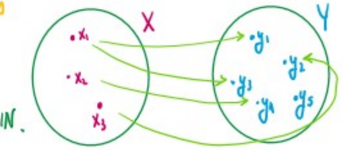


Ex. 1



Yes, this assignment is a fct.

Ex. 2



Not a function b/c 2 y-values for 1 x_1.

(ii) $f: \mathbb{N} \longrightarrow \mathbb{Z}$, $f(n) = n^2$. is a function. Note that $-3 \neq f(n)\ \forall n \in \mathbb{N}$.

Rmk: we could do $f: \mathbb{Z} \to \mathbb{Z}$, $f(m) = m^2$

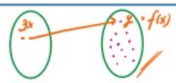
(iii) $f: \mathbb{N} \longrightarrow \mathbb{Z}$,

$f(1) = 0,\ f(3) = -1,\ f(5) = -2,\ f(7) = -3$

$f(2) = 1,\ f(4) = 2,\ f(6) = 3,\ f(8) = 4$

$\left.\right\}$ $\boxed{f(2n) = n,\ f(2n+1) = -n}$ defines $f: \mathbb{N} \longrightarrow \mathbb{Z}$

§2. Three properties :



Def^n: • (SURJECTIVE) A function $f: X \longrightarrow Y$ is said to be onto (surjective or epi, epimorphism) ← equiv. words

if $\boxed{\forall y \in Y\ \exists x \in X \text{ s.t. } f(x) = y}$

• (INJECTIVE) A function $f: X \longrightarrow Y$ is said to be 1-to-1 (injective or mono, monomorphism) ← equiv. words

if $\boxed{\forall x_1, x_2 \in X \text{ s.t. } f(x_1) = f(x_2) \text{ then } x_1 = x_2}$ ⇝ i.e. two different $x_1, x_2 \in X$ must have $f(x_1) \neq f(x_2)$.

• A function is bijective if it is $\boxed{\text{INJECTIVE AND SURJECTIVE}}$.

Def^n: X, Y sets have the same cardinality if $\exists f: X \longrightarrow Y$ bijection

↗ here, X, Y might have "∞" many elements

Examples : (i) $f: \mathbb{N} \longrightarrow \mathbb{Z}$, $f(n) = n^2$. Injective, not surjective

$-3 \in \mathbb{Z}$, $-3 = f(n) = n^2$

↳ $\nexists n \in \mathbb{N}$ s.t. $f(n) = -1$. b/c $f(n) \geq 0$.

(ii) $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(n) = n^2$. Not injective, not surjective.

↳ e.g. $x_1 = -1$, $x_2 = 1$ $f(1) = f(-1)$ as both square $(-1)^2 = 1^2 = 1$.

(iii) $f: \mathbb{N} \longrightarrow \mathbb{Z}$,

$f(k) = \begin{cases} n & \text{if } k = 2n \\ -n & \text{if } k = 2n+1 \end{cases}$ ⇝ $\boxed{\text{INJECTIVE and SURJECTIVE}}$

proof

Thm : $\exists f: \mathbb{N} \longrightarrow \mathbb{Z}$ bijection. In particular, $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$.

$\exists$ many different f's, we exhibited one ✓

new !

What's next ? $\text{card}(\mathbb{Q})$? $\text{card}(\mathbb{R})$? $\text{card}(\mathbb{I})$? irrationals