Lecture 22

Tuesday, September 29, 2020 12:05 PM

Lecture 22 : Countable & Uncountable Sets

→ card($\mathbb{N}$) = card($\mathbb{Z}$) b/c we built bijection $\mathbb{N} \to \mathbb{Z}$.

Def^n: Let $[n] := \{1,2,3,\dots,n\}$, and X a set.

(i) X is finite if $\exists n \in \mathbb{N}$ s.t. X bijects to $[n]$.

(ii) X is countably infinite if card(X) = card$(\mathbb{N})$. ← i.e. $\exists$ bijection between X and $\mathbb{N}$

X is countable if X is finite or countably finite. ← (e.g. $\mathbb{Z}$ is countable, in fact countably infinite)

(iii) X is uncountable if it is not countable.

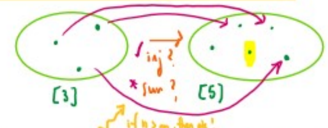
⟶ is $\mathbb{Q}$ countable or not? What about $\mathbb{R}$? What about $\mathbb{R} \setminus \mathbb{Q}$? (complement = irrationals)

§1. Finite sets

Thm. (13.4) If $m, n \in \mathbb{N}$ with $n \neq m$. Then $[n]$ does not biject to $[m]$.

e.g. $[3] = \{1,2,3\}$, $[5] \cong \{a,b,c,d,e\}$

Exercise: Prove thm. by induction. (See textbook.)



Thm. (13.5 - "Pigeonhole Principle") If $m, n \in \mathbb{N}$ s.t. $n < m$. Then any function $f : [m] \longrightarrow [n]$ cannot be injective.

Rmk: Similarly, no map from $[n] \to [m]$ can be surjective.

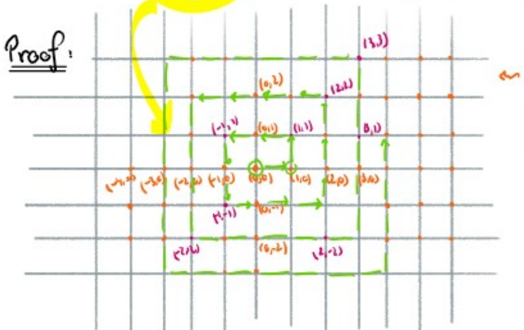
E.g.

§2. Countable infinite

$\mathbb{N}$ is countably infinite, by Friday $\mathbb{Z}$ is countably infinite.

Thm. (13.14): $\mathbb{Z} \times \mathbb{Z}$ is countable. ($\Rightarrow$ COROLLARY: $\mathbb{Q}$ is countable.) b/c $r \in \mathbb{Q}$ is $r = \frac{n}{m}$, $n, m \in \mathbb{Z}$, $m \neq 0$

Proof:



↜ visualizing $\mathbb{Z} \times \mathbb{Z} \overset{?}{=} \{(n,m) : n, m \in \mathbb{Z}\}$

Spiralling out establishes a bijection*

$\mathbb{N} \longrightarrow \mathbb{Z} \times \mathbb{Z}$

$k \longmapsto (n,m)$

point in grid that we hit after k steps!

* every point in grid is visited EXACTLY ONCE!

Properties for countable sets

: Let X, Y be countable sets. (e.g. $X = \mathbb{N}$, $Y = \mathbb{Q}$ or finite sets too)

(a) $\boxed{X \times Y \text{ is countable}}$ (same proof as Thm. 13.14 b/c $X \cong \mathbb{N}$ or $\mathbb{Z}$, $Y \cong \mathbb{N}$ or $\mathbb{Z}$)

(b) If $A \subseteq X$ is a subset, A must be countable. ↗ equiv: if $X \cup Y$ uncountable, either X or Y are uncountable

More general: the union $\boxed{X \cup Y \text{ is countable}}$ (in fact countable union is countable)

Examples: (i) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ the set of triples of $\mathbb{Q}$-numbers is COUNTABLE.

(ii) The set of integer sequences $\{X_n : n \in \mathbb{N}, X_n \in \mathbb{Z}\}$ is countable.

morally the same ← $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z} \cup \dots \cup \mathbb{Z} \cup \dots^{(\mathbb{N})}$ are countable.