Lecture 23

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Lecture 23 : Uncountable Sets : $\mathbb{R}$, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ ~ irrationals

same cardinality! | count. or uncount.? | complement

Theorem : $\mathbb{R}$ is an uncountable.

↳ today's goal!

we need to show that $\nexists$ $f : \mathbb{N} \to \mathbb{R}$ bijection ~ will do: by contradiction

(*) $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| =: \aleph_0$. ← smallest infinity!
$|\mathbb{R}| =: \aleph_1$. There are (∞) more, e.g. $\mathbb{R}^{\mathbb{R}}$.

Corollary : The irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$) are an uncountable.

Proof of Corollary : If A, B are countable, then $A \cup B$ is countable. Therefore, if $A \cup B$ is uncountable, it must be that A is uncountable or B is uncountable.

Choose $A = \mathbb{I}$, $B = \mathbb{Q}$, then $A \cup B = \mathbb{R}$ is uncountable by Thm. Since $\mathbb{Q}$ count., $\mathbb{I}$ uncountable. ∎

§2. Cantor's diagonal argument : $\mathcal{B}$ = set of binary seq = $(x_n)_{n \in \mathbb{N}}$ s.t. $x_n = 0$ or 1.

E.g. $0000\ldots0\ldots \in \mathcal{B}$, $1111\ldots1\ldots \in \mathcal{B}$, $011010001\ldots \in \mathcal{B}$ are elements.

intuitively, $\mathcal{B} = 2^{\mathbb{N}}$, i.e. all possible subsets of $\mathbb{N}$.

Thm : $\mathcal{B}$ is an uncountable set.

Proof : Suppose that $\exists$ bijection $f : \mathbb{N} \longrightarrow \mathcal{B}$. so $f(n)$ is a binary seq.

1 2 3 4 …

some bin. seq.

$$
\begin{array}{lccccccccccl}
f(1) = & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \ldots \\
f(2) = & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \ldots \\
f(3) = & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
f(4) = & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & \ldots \\
f(5) = & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & \ldots \\
\vdots & & & & & & & & & & & \\
f(n) = & \text{a bin. seq.} & & & & & & & & & &
\end{array}
$$

this is the DIAGONAL!

Given this, consider the binary seq. $s \in \mathcal{B}$ as follows

$s = 0 \quad 1 \quad 0 \quad 1 \quad 0 \ldots s_n \ldots$,

look at 1st term in $f(1)$, put the opposite

$(s \neq f(n), \forall n \in \mathbb{N})$

NOT of the form $f(n)$ $\forall n \in \mathbb{N}$ $\Rightarrow$ f not surjective!

the opposite of the n^{th} term in $f(n)$

CONTRADICTION! ∎

§3. Proof of Main Thm : $\mathbb{R}$ is uncountable.

((*) One can prove that $|\mathcal{B}| = |\mathbb{R}|$ by building a bijection.)

(i) First, we note that if A is countable and $f : T \longrightarrow A$ is an injection, then T countable.

Stated otherwise : if T is uncountable and $f : T \to A$ injection, then A is uncountable.

(ii) $\exists$ an injection $i : \mathcal{B} \longrightarrow \mathbb{R}$ (we will not need it to be surjection!), defined by :

$$i(b_1 b_2 b_3 b_4 \ldots b_n \ldots) := 0.b_1 b_2 b_3 b_4 \ldots b_n \ldots$$

decimal expansion for real number $r \in \mathbb{R}$

E.g.: $i(000\ldots0\ldots) = 0.000\ldots0 = 0 \in \mathbb{R}$

$i(1000\ldots0\ldots) = 0.1000\ldots0\ldots = 0.1 \in \mathbb{R}$

$i(10101010\ldots10\ldots) = 0.1010101010\ldots10\ldots \in \mathbb{R}$

by (i), since $\mathcal{B}$ is uncountable ($T = \mathcal{B}$, $A = \mathbb{R}$) and $i : \mathcal{B} \to \mathbb{R}$ injection, we have $\mathbb{R}$ uncountable! ∎