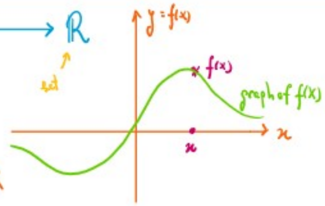
Lecture 24

Tuesday, September 29, 2020 12:05 PM

# Lecture 24 : Continuity of functions , $f : \mathbb{R} \longrightarrow \mathbb{R}$

function set set




**Example** : (i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

try to draw its graph

jumps everywhere

$\leadsto$ continuous ? ✗

(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$  jump  $\leadsto$ continuous ? ✗

(iii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) =$ polynomial in $x$  polynomial  $\leadsto$ continuous ? ✓

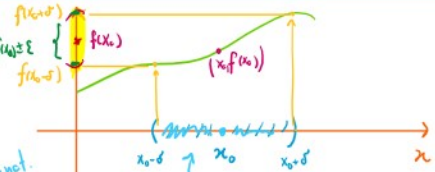
What is happening at a jump ? :

y = f(x), f(x₀), small if δ is small, $x_0 - \delta$, $x_0$

y = f(x), f(x₀), f(x₀ - δ), jump !, $x_0 - \delta$, $x_0$, small

## §2. The ε–δ definition :

small interval near $x_0$ gives interval around $f(x_0)$ ↑ might be small or not. (that's continuity)




**Def** : $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0$ if $\forall \varepsilon > 0$

$\exists \delta > 0$ such that

$|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \varepsilon$ ∎

happens in f(x) axis

in x-axis

**Two examples** : (i) $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$ is not continuous at 0.

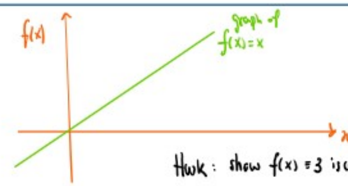
f(x), 1, y, e.g. $y = \delta/2$, $-\delta$, $\delta$, x

$\exists \delta$ ? Try to find it !

"X" gives you $\varepsilon = 1/4$. Take $x_0 = 0$, $y$ s.t. $|x_0 - y| < \delta$, i.e. $|y| < \delta$. We can take $y = \delta/2$. satisfies

Now $f(x_0) = 0$, $f(y) = f(\delta/2) = 1$, hence $|f(x_0) - f(y)| = |0 - 1| = 1 \nless \varepsilon = 1/4 \Rightarrow \delta$ does not exist !

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x$.




Claim : given $\varepsilon > 0$, we need $\delta > 0$ s.t.

(*) $\boxed{|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon}$

choice of $\delta$ depends on $\varepsilon$ !!!

Hwk : show $f(x) = 3$ is continuous, show $f(x) = x^2$ is continuous. ↖ exciting !

Since $f(x) = x$, $f(y) = y$, $\boxed{|x - y| < \delta \Rightarrow |\underset{f(x)}{x} - \underset{f(y)}{y}| < \varepsilon}$ *, how to choose $\delta$ ?

In this case, it suffices to choose $\delta := \varepsilon$, then Box (*) reads

$\boxed{|x - y| < \varepsilon \Rightarrow |x - y| < \varepsilon}$, which is true. $\Rightarrow f$ continuous at all points !