

# Lecture 3: PRINCIPLE OF MATHEMATICAL INDUCTION

we introduced  $\mathbb{N} \subseteq \mathbb{Z}$  <sup>integers</sup> with them we can introduce an order as follows:  
nature a subset

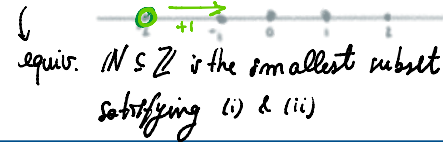
Def: Let  $n, m \in \mathbb{Z}$ . we write  $m < n$  if  $n + (-m) \in \mathbb{N}$ .  $\square$   
 $n > m$  equiv.  $n - m \in \mathbb{N}$

Corollary:  $\mathbb{N} = \{n \in \mathbb{Z} \text{ s.t. } n > 0\}$ . (equiv.  $n - 0 \in \mathbb{N}$ )  $\circledast$  Proving = between sets means  $\subseteq$  and  $\supseteq$ .

Work through properties of the order " $<$ " defined above, in particular prove Prop. 2.4 through 2.13.

$\S 1$ . Induction: a method of proof useful: Pset 2 has tons of problem solvable by induction  
 we add one last axiom for  $\mathbb{N} \subseteq \mathbb{Z}$ :

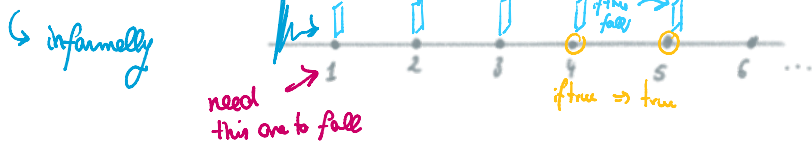
Axiom 2.15: if a subset  $A \subseteq \mathbb{Z}$  satisfies  
 (i)  $1 \in A$  (ii) if  $n \in A$  then  $n+1 \in A$  } assumption  
 Then  $\mathbb{N} \subseteq A$ . conclusion



Ex:  $A = \mathbb{N} \cup \{0\}$  has (i) & (ii)  
 $A = \mathbb{N} \cup \{-1, 0\}$  has (i) & (ii).

proof avoided The main application of Axiom 2.15 is:

Theorem 2.17. (Principle of induction) Let  $P(n)$  be a statement that depends on  $n \in \mathbb{N}$ . If  
 (1)  $P(1)$  is true base case  
 (2)  $\forall n \in \mathbb{N}, P(n)$  true implies  $P(n+1)$  true induction step  
 Then  $P(n)$  is true  $\forall n \in \mathbb{N}$ .  
if one piece falls next falls



Example of a problem:  $P(n) = "7 \text{ divides } 5^{2n+1} + 2^{2n+1} \text{ for all } n \in \mathbb{N}."$

(also 2.18 in book & Pset 2)  
 $n=1: P(1) = "7 \text{ divides } 5^3 + 2^3."$  statement depends on  $n \in \mathbb{N}$ .  
 $n=2: P(2) = "7 \text{ divides } 5^5 + 2^5."$   
 $n=3: P(3) = "7 \text{ divides } 5^7 + 2^7."$  and so on.

Two steps for induction are:  
 (i) Base case  
 (ii) Induction step.

Proof: By induction. The base case is  $P(1) = "7 \text{ divides } 5^3 + 2^3"$ , since  $5^3 + 2^3 = 133 = 7 \cdot 19$ , the statement  $P(1)$  is true. For the induction step, we assume  $P(n)$  is true and try to deduce  $P(n+1)$  is true. We assume " $7 \text{ divides } 5^{2n+1} + 2^{2n+1}$ ", we want to show that " $7 \text{ divides } 5^{2n+3} + 2^{2n+3}$ " =:  $P(n+1)$  is true.