Lecture 4

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Lecture 4 : Principle of Mathematical Induction II

Thm 2.17 : (induction) Let $P(n)$ be a statement depending on $n \in \mathbb{N}$.

assumption: If (i) $P(1)$ is true, ← base case

(ii) $P(n)$ being true implies $P(n+1)$ is true. ← induction step

concl. Then $P(n)$ true for all $n \in \mathbb{N}$.

Examples: $P(n)$ could be a divisibility statement (see Prop. 2.18),

it could be a CLOSED formula for sum, also inequalities between $n \in \mathbb{N}$ (see Prop 2.24)

e.g. $1+2+3+\dots+n = \frac{n(n+1)}{2}$ ← closed expr. ⟶ check Problem Set 2

Proof of Thm 2.17 : We want $P(n)$ true for all $n \in \mathbb{N}$. ← equiv.

Axiom we use: The Induction Axiom A2.15 tells us that a subset $B \subseteq \mathbb{Z}$ (is a subset of) s.t.

(i) $1 \in B$ (ii) if $b \in B$ then $b+1 \in B$

then $\mathbb{N} \subseteq B$.

Consider the set $\boxed{B := \{ n \in \mathbb{N} \text{ s.t. } P(n) \text{ is true} \}}$ (defined to be), we want that $B = \mathbb{N}$.

First, note $B \subseteq \mathbb{N}$. Thus it suffices to show $\mathbb{N} \subseteq B$. For that we check if B satisfies (i) & (ii). Is (i) satisfied? $1 \in B \iff P(1)$ true which is true by assumption.

Is (ii) satisfied? $n \in B$ imply $n+1 \in B$??? $\iff$ $P(n)$ true implies $P(n+1)$ true? True by assumption.

$\implies \mathbb{N} \subseteq B \implies \mathbb{N} = B \iff P(n)$ true for all $n \in \mathbb{N}$ ∎

Problem 1 : Show that $\forall n \in \mathbb{N}$ $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

Sol^n (solution): By induction on $n \in \mathbb{N}$.

STEP 1 : Base case $n=1$: for $n=1$ this $5^{2\cdot 1+1} + 2^{2\cdot 1+1} = 5^3 + 2^3 = 125 + 8 = 133$ divisible by 7. (true) ✓ ok!

STEP 2 : Induction step: Suppose the statement is true for n. Is it true for $n+1$?

suppose $7 \mid 5^{2n+1} + 2^{2n+1}$ want $7 \mid 5^{2(n+1)+1} + 2^{2(n+1)+1}$

We start with

$7 \overset{?}{\mid} 5^{2n+3} + 2^{2n+3} \iff 7 \overset{?}{\mid} 25 \cdot 5^{2n+1} + 4 \cdot 2^{2n+1} \iff 7 \overset{???}{\mid} 21 \cdot 5^{2n+1} + 4 (5^{2n+1} + 2^{2n+1})$ ⇑ TRUE

($7 \mid$ divides; $25 = 21 + 4$)

Since $21 = 7 \cdot 3$ then $7 \mid 21 \cdot (\text{anything})$ ($7 \mid 21$), also by assumption $\boxed{7 \mid 5^{2n+1} + 2^{2n+1}}$ ⟹ 7 divides each SUMMAND ⟹ 7 divides the sum ∎

Problem 2 : $\boxed{\forall n \in \mathbb{N}, \ \sum_{k=1}^{n} \underbrace{(2k-1)}_{\text{summand}} = n^2 .}$

sum from $k=1$ to $k=n$

sum all the first n odd natural numbers

- Examples:

base case $n=1$: $\sum_{k=1}^{n=1} (2k-1) = 1 = n^2$ when $=1$

$n=2$: $1+3 = 4$

$n=3$: $1+3+5 = 9$ ← squares!

$n=4$: $1+3+5+7 = 16$

first 100^{th} odd number

$n = 100$? : $1+3+5+7+9+\dots+97+99+\dots+199 = 100^2 = 10{,}000$.

how to prove it?

By induction!

(base case $n=1$ ✓)