Lecture 5

Tuesday, September 29, 2020 12:05 PM

Lecture 5 : Problems & Applications ← today we finish INDUCTION. (RECURSION WEEK NEXT!)

DIVISIBILITY

(i) $\boxed{7 \mid 5^{2n+1} + 2^{2n+1} \quad \forall n \in \mathbb{N}}$

(ii) See Prop. 2.18 in textbook

(iii) $a+b \mid a^{2n+1} + b^{2n+1} \quad \forall n \in \mathbb{N}$.

SUMS & CLOSED FORMULAE

(i) $\boxed{\sum_{i=1}^{n} (2i-1) = n^2}$. sum of first n odd numbers

(ii) $\sum_{i=1}^{n} 2i = n(n+1)$

also see PSet 2: $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$? (by induction!)

INEQUALITIES

(i) See Recitation on Thu Oct 8

(ii) $\boxed{\forall n \in \mathbb{N},\ 2^n \leq n! \quad n \geq 4}$ ← base case is $n=4$!

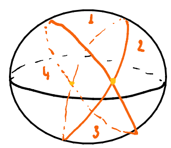
(iii) See textbook Prop. 2.24 & 2.27 and PSet 2.

GEOMETRIC PROBLEMS

(i) See Recitation Oct 8

(ii) Prob. 6&7 in PSet 2 ← by induction

(iii)



how many REGIONS if we use n distinct ⊘ ?

Problem 1 : Show that $\sum_{i=1}^{n} (2i-1) = n^2, \ \forall n \in \mathbb{N}$.

Sol^n: By induction on $n \in \mathbb{N}$. Two steps:

(1) BASE CASE: this is $n=1$, and the formula reads $\sum_{i=1}^{1} (2i-1) = 1^2 \Leftrightarrow 1=1$ which is true. ($2\cdot 1 - 1 = 1$ ✓)

(2) INDUCTION STEP: we assume $\boxed{\sum_{i=1}^{n} (2i-1) = n^2}$ (*), we want $\boxed{\sum_{i=1}^{n+1} (2i-1) = (n+1)^2}$. (#) n+1 case

Indeed,

$$\sum_{i=1}^{n+1} (2i-1) = \left[\sum_{i=1}^{n} (2i-1) + 2(n+1)-1 \right] \underset{\uparrow}{=} n^2 + 2n+1 = (n+1)^2 \quad \text{as required.} \ \blacksquare$$

term of $i=n+1$; n case; (*) induction assumption is used

$[1+3+5+\dots+11+\dots+(2n-1)] + (2(n+1)-1)$, $i=n+1$; $1+3+\dots+2n-1$

Problem 2 : Show that $\boxed{2^n \leq n!}$ is true for $\forall n \in \mathbb{N}$, $n \geq 4$.

$n! = 1\cdot 2 \cdot 3 \cdot 4 \cdots n$, e.g. $1!=1$, $2!=2$, $3!=6$, $4!=24$, $5!=120$

$(n+1)! = n!\,(n+1)$.

Sol^n: By induction, we need to verify 2 steps:

(1) BASE CASE: This is $n=4$, need to check $2^4 \leq 4!$. Since $2^4 = 16$ and $4! = 4\cdot 3\cdot 2\cdot 1 = 24$ and $16 \leq 24$, this is true.

(2) INDUCTION STEP: we assume $\boxed{2^n \leq n!}$ (*), we want $\boxed{2^{n+1} \leq (n+1)!}$. RHS $= n!\,(n+1)$

We start with

$$2^{n+1} = 2\cdot 2^n \underset{(*)}{\leq} 2\cdot n! \leq (n+1)\cdot n! = (n+1)! \quad \text{as required} \ \blacksquare$$

use $2 \leq n+1$ since $n \geq 4$

$n=4$: $2^4 \leq 4!$ ✓ (16, 24)

$n=5$: $2^5 \leq 5!$ ✓ (32, 120)

$n=6$: $2^6 \leq 6!$ ✓ (64, 720)

Problem 3 : (Euclid) ∃ infinitely many primes.

Sol^n: By contradiction, assume the opposite: "∃ finitely primes". ⇝ try to reach a contradiction

If we have finitely many primes, then we write them as $\boxed{p_1, p_2, p_3, \dots, p_s}$. ($p_i \neq 1$) allegedly all primes

Now consider $\{ P := p_1 p_2 p_3 \cdots p_s + 1 \}$. ⟶ $1 = P - p_1 \cdots p_s$

Since $P > p_i$, $i=1,\dots,s$, $P \neq p_i$. So P cannot be a prime. It must be that $\exists p_i$ (Not prime) s.t. $p_i \mid P$. But $p_i \mid p_1 \cdots p_s$ (contains p_i). If $p_i \mid P$, $p_i \mid p_1 \cdots p_s$ then $p_i \mid 1$. CONTRADICTION! $\blacksquare$