Lecture 6

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Lecture 6 : Introduction to recursion

RECURSION WEEK! (Chap. 4)

A sequence $\{x_j\}_{j\in\mathbb{N}}$ is a list of integers indexed by the set $\mathbb{N}$.

Example: (i) $\{x_j\} = (x_1, x_2, x_3, x_4, \ldots, x_{17}, x_{18}, \ldots, x_{523}, x_{524}, \ldots)$ ← general form

(ii) $\{x_j\} = (7, -23, 4, 5, 6, -81, 7, 22, -34, \ldots)$ ↔ x_j chosen randomly

(iii) $\{x_j\} = (1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots)$ ↔ x_j term can described as $x_j := 2\cdot x_{j-1}$ — also we could see $x_j = 2^{j-1}$.

(iv) $\{x_j\} = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots)$ RECURSION

term of seq. depend via formula on previous one — closed formula for j^{th} term

Fibonacci sequence $x_j = x_{j-1} + x_{j-2}$.

Three classes of recursion : $\prod$, $\sum$ and !'s. (Section 4.1)

(1) A sequence $(s_1, s_2, s_3, \ldots, s_k, \ldots)$. Define $\{x_j\}$ s.t. $x_j := \sum_{i=1}^{j} s_i$. — how sums $\sum_{i=1}^{j} s_i$ are defined

In particular, $\boxed{x_{j+1}} := \sum_{i=1}^{j+1} s_i = \left(\sum_{i=1}^{j} s_i\right) + s_{j+1} = \boxed{x_j + s_{j+1}}$

(2) $\{x_j\}$ defined by $\boxed{x_{j+1} = x_j\cdot(j+1)}$, with start $x_1 = 1$.

$\{x_j\} = (1, 1\cdot2, 1\cdot2\cdot3, 1\cdot2\cdot3\cdot4, 1\cdot2\cdot3\cdot4\cdot5, \ldots) = (1, 2, 6, 24, 120, 720, \ldots)$

$\underbrace{1}_{x_1}$, $\underbrace{1\cdot 2}_{x_2}$, $\underbrace{1\cdot2\cdot3}_{x_3}$

Def: The term x_j in this sequence is called j-factorial, denoted $j!$. — appear a lot on wednesday

§2. Geometric series

Fix a number $r = 3$ (ratio). Consider $\{s_i\} = (1, 3, 9, 27, 81, \ldots)$ ($\times3$, $\times3$, $\times3$, $\times3$)

How do we sum the first n^{th} terms of this of sequence?

$1 = 1$
$1+3 = 4$
$1+3+9 = 13$
$1+3+9+27 = 40$
$1+3+9+27+81 = 121$

$3^0+3^1+3^2+\ldots+3^7 = ?$

$3^0+3^1+3^2+\ldots+3^n = ?$

Def: A sequence $x_j := \sum_{i=0}^{j} r^i$ is called a geometric series.

Challenge: Let's find a closed formula

$1 + r^1 + r^2 + r^3 + r^4 + \ldots + r^n$ ↔ $\cdot 1$

$-\; r + r^2 + r^3 + r^4 + r^5 + \ldots + r^{n+1} = (1+r^1+r^2+r^3+r^4+\ldots+r^n)\,r$, thus

$1 \quad - r^{n+1}$

$$\left(\sum_{i=0}^{n} r^i\right)(1-r) = 1-r^{n+1} \iff \boxed{\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}}$$

Prop. 4.13: For $r \neq 1$, $n\in\mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$.

Proof: By induction, there are two steps:

(i) Base Case: We want $\sum_{i=0}^{0} r^i = \frac{1-r}{1-r}$, this is $r^0 = 1$, which is true.

(ii) Inductive step: We assume $\boxed{\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}}$, we want $\sum_{i=0}^{n+1} r^i = \frac{1-r^{n+2}}{1-r}$. Indeed,

$$\sum_{i=0}^{n+1} r^i = \left(\sum_{i=0}^{n} r^i\right) + r^{n+1} = \frac{1-r^{n+1}}{1-r} + r^{n+1} = \frac{1-r^{n+1} + (1-r)r^{n+1}}{1-r} = \frac{1-r^{n+2}}{1-r} \qquad \blacksquare$$