

Lecture 7: The Binomial Theorem (Sec. 4.4)

Def: Given $n \in \mathbb{N}$, we define $n! := (n-1)! \cdot n$, with $1! = 1$.
 In practice: $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$.

e.g. $3! = 6$, $4! = 24$
 $5! = 120$

Combinatorially: how many ways can we order elements from a set of size n ? $3! = 6$
 size $n=3$ $X = \{\Delta, O, \square\}$, orders Δ, O, \square or O, Δ, \square or \square, Δ, O or \square, O, Δ or Δ, \square, O or O, \square, Δ

Def: Given $k, n \in \mathbb{N}$, $k \leq n$. We define $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ ← "n choose k"
 $n!$ = orders for n balls
 $k!$ = orders for k balls
 $(n-k)!$ = orders for $n-k$ balls

Combinatorially: how many ways can we choose k elements out of n elements?
 $X = \{12 \text{ kids } \{ \}$, we want a team of 5, what choices? $\binom{12}{5} = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{120 \cdot 7!} = 11 \cdot 9 \cdot 8 = 99 \cdot 8 = 792$.

§ 3. Binomial Theorem:

Thm. 4.21: let $x, y, n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} \cdot x^{n-k} \cdot y^k$$

(convention) $0! = 1$

$k=0, 1, 2, \dots, n$
 $k \in \mathbb{N} \cup \{0\}$

Example: $n=4$, then we need to compute $\binom{4}{k}$, $k=0, 1, 2, 3, 4=n$.

$$\frac{4!}{0!4!} = \binom{4}{0} = 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = \frac{4!}{2!2!} = 6$$

$$\binom{4}{3} = 4, \quad \binom{4}{4} = 1$$

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

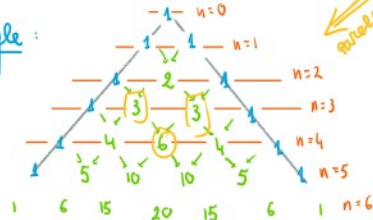
Tells you THE COEFFICIENTS!

§ 2. Expansions of $(x+y)^n$

For $n=1$, $(x+y)^1 = x+y$
 $n=2$, $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$
 $n=3$, $(x+y)^3 = (x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$
 $n=4$, $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Cor. 4.20: For $k, n \in \mathbb{N}$ try by induction
 $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
 why is this true?
 LHS: choose k out of n
 RHS: ? → choose not FLAG PINK

Pascal's triangle:



$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Proof of Thm 4.21: we could do it by induction - see textbook - but we argue directly, as follows:

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y) \dots (x+y)(x+y)}_{n \text{ terms}} = 1 \cdot x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{k} x^{n-k} y^k + \dots + \binom{n}{n-1} x y + \binom{n}{n} y^n$$

n y's here
 n y's here
 we want k y's to get y^k } $\binom{n}{k}$ ways to do so