Lecture 7

Tuesday, September 29, 2020 12:05 PM

Lecture 7: The Binomial Theorem (Sec. 4.4)

Def: Given $n \in \mathbb{N}$, we define $n! := (n-1)! \cdot n$, with $1! = 1$.

In practice: $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

e.g. $3! = 6$, $4! = 24$, $5! = 120$

Combinatorially: how many ways can we order elements from a set of size n? $3! = 6$

size $n=3$: $X = \{\Delta, O, \square\}$, orders: $\Delta, O, \square$ or $O, \Delta, \square$ or $\square, \Delta, O$ or $\square\ O\ \Delta$ or $\Delta\ \square\ O$ or $O\ \square\ \Delta$

Def: Given $k, n \in \mathbb{N}$, $k \le n$. We define $\binom{n}{k} := \dfrac{n!}{k!\,(n-k)!}$ "n choose k"

$n!$ = orders for n elts: $\underbrace{\ }_{k!\ :3!}\ \underbrace{\ }_{(n-k)!}$ $\quad \frac{7!}{3!\,4!}$

Combinatorially: how many ways can we choose k elements out of n elements?

$X = \{12 \text{ kids}\}$, we want a team of 5, what choices? $\binom{12}{5} = \dfrac{12!}{5!\,7!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{120 \cdot 7!} = 11 \cdot 9 \cdot 8 = 99 \cdot 8 = 792$.

§2. Expansions of $(x+y)^n$

For $n=1$, $(x+y)^1 = x+y$

$n=2$, $(x+y)^2 = (x+y)(x+y) = 1x^2 + 2xy + 1y^2$

$n=3$, $(x+y)^3 = (x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$

$n=4$, $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

of Prop. 4.19 in textbook

Cor. 4.20: For $k, n \in \mathbb{N}$ — try by induction

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

row in Pascal's triangle

why is this true?

LHS: choosing k out of n ✓

RHS: ? → choose wrt FLAG PERSON

Pascal's triangle:

n=0: 1

n=1: 1 1

n=2: 1 2 1

n=3: 1 3 3 1

n=4: 1 4 6 4 1

n=5: 1 5 10 10 5 1

n=6: 1 6 15 20 15 6 1

reason: parallel arrows

$(x+y)^6 = 1 \cdot x^6 + 6 \cdot x^5y + 15\,x^4y^2 + 20 \cdot x^3y^3 + 15 \cdot x^2y^4 + 6 \cdot xy^5 + 1 \cdot y^6$

§3. Binomial Theorem:

Thm. 4.21: Let $x, y, n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot y^k \quad \blacksquare$$

"n choose k"

$k = 0, 1, 2, \ldots, n$, $k \in \mathbb{N} \cup \{0\}$

TELLS YOU THE COEFFICIENTS!

(convention $0! = 1$)

Example: $n=4$, then we need to compute $\binom{4}{k}$, $k = 0, 1, 2, 3, 4 = n$.

$\frac{4!}{0!\,(4-0)!} = \binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$

$n=4$: $(x+y)^4 = \binom{n}{0}x^4 + \binom{n}{1}x^3y + \binom{n}{2}x^2y^2 + \binom{n}{3}xy^3 + \binom{n}{4}y^4$

$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Proof of Thm. 4.21: we could do it by induction — see textbook — but we argue directly, as follows:

$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)(x+y)}_{n \text{ terms}} = 1 \cdot x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots$

n y's in here

$+ \binom{n}{k} x^{n-k} \cdot y^k + \ldots + \binom{n}{n-1} \cdot xy^{n-1} + \binom{n}{n} y^n$

$\underbrace{(x+y)(x+y)(x+y)\cdots(x+y)(x+y)}$

n y's here, we want k y's to get y^k } $\binom{n}{k}$ ways to do so

$\blacksquare$