# Lecture 8

Tuesday, September 29, 2020    12:05 PM

## Lecture 8 : Homogeneous Linear Recursions ← not in textbook!

**Def:** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence, then $(x_n)_{n\in\mathbb{N}}$ is defined by a linear recursion if $\left(x_n := \sum_{i=1}^{n-1} \alpha_i x_i = \alpha_{n-1}x_{n-1} + \alpha_{n-2}x_{n-2} + \ldots + \alpha_2 x_2 + \alpha_1 x_1\right)$ $\alpha_i \in \mathbb{Z}$

$x_n := 3\cdot x_{n-1} - 7x_{n-2} + \ldots + 4x_2 - 11\cdot x_1$ ($\alpha_{n-1}$, $\alpha_{n-2}$, $\alpha_2$, $\alpha_1$)

sum (addition) for linear, products are not allowed (at all)

**Ex:** (i) Sequence $\boxed{x_n = x_{n-1} + x_{n-2}}$ with $x_1 = x_2 = 1$. LINEAR as in

Then $(x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$ ← Fibonacci seq.

same recursion but different initial values

(ii) Sequence $\boxed{x_n = x_{n-1} + x_{n-2}}$ with $x_1 = 1, x_2 = 3$.

Then $(x_n) = (1, 3, 4, 7, 11, 18, \ldots)$ ← Lucas seq.

NOT LINEAR (BUT A RECURSION)

(iii) Sequence $\boxed{x_n = x_{n-1}\cdot x_{n-2}}$, $x_1 = x_2 = 2$, then $(x_n) = (2, 2, 4, 8, 32, \ldots)$

e.g. $x_{n-1}\cdot x_{n-2} + x_3$ not linear

## §1. Goal : Finding a closed formula

never having to compute the terms $x_{n-1}, x_{n-2}$

**Ex:** $(x_n) = (1,1,2,3,5,\ldots)$ Fibonacci, i.e. $x_n := x_{n-1} + x_{n-2}$ w/ $x_1 = x_2 = 1$.

Given the terms up to $x_{n-1}$, finding $x_n$ is just. The challenge is to find a general formula for $x_n$, i.e. $x_n =$ "formula only in terms of $n$".

Today, we will study a general formula for recursions of the form :

$$(\#) \quad \boxed{x_n := A\cdot x_{n-1} + B\cdot x_{n-2}}, \quad A, B \in \mathbb{Z}.$$

use a new technique called "characteristic polynomials".

## §2. Characteristic Polynomial : given the recursion (#) $x_n := A\cdot x_{n-1} + B\cdot x_{n-2}$,

**Def:** The char. polynomial of (#) is $p(r) = r^2 - Ar - B$.
Its roots are denoted $r_1$ & $r_2$, and we'll assume $r_1 \neq r_2$.

$x_n \to r^2$, $x_{n-1} \to r$, $x_{n-2} \to r^0$

linear recursion / recurrence only using previous two terms

**Thm:** Let $(x_n)$ be a recursive with $x_n := Ax_{n-1} + Bx_{n-2}$.
Consider $p(r) := r^2 - Ar - B$ and its roots $r_1, r_2$ w/ $r_1 \neq r_2$. (these are known)
Then,
$$x_n = C\cdot r_1^n + D\cdot r_2^n,$$
with $C, D \in \mathbb{R}$ determined by the values $x_1, x_2$.

Ex: How to compute $C, D \in \mathbb{Z}$?
Plug the known values of $x_1, x_2$
e.g. if $x_1 = 3$, $x_2 = -4$

$$\left.\begin{aligned} 3 &= C\cdot r_1 + D\cdot r_2 \\ -4 &= C\cdot r_1^2 + D\cdot r_2^2 \end{aligned}\right\} \text{ solve for } C, D.$$

## §3. The Fibonacci Sequence & Binet's Formula : let $f_n := f_{n-1} + f_{n-2}$ w/ $f_1 = f_2 = 1$. ($A=1$, $B=1$)

Then $(f_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots)$. Can we find formula for $f_n$?

**Sol$^n$:** By Thm., we need the roots $r_1, r_2$ of the characteristic polynomial :

$p(r) = r^2 - Ar - B \overset{A=B=1}{=} r^2 - r - 1$, they are $r_1, r_2 = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

We choose $r_1 := \frac{1+\sqrt5}{2} = 1.618\ldots$, $r_2 := -0.618\ldots = \frac{1-\sqrt5}{2}$. Then we get

$$f_n \overset{\text{Thm}}{=} C\left(\frac{1+\sqrt5}{2}\right)^n + D\cdot\left(\frac{1-\sqrt5}{2}\right)^n \longrightarrow \boxed{f_n = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right]}$$

RHS can be computed directly plug in $n$ !!

need to find $C$ & $D$ ← use

(*) $\boxed{\begin{aligned} &f_1 = 1 \text{ so } 1 = C\cdot\left(\tfrac{1+\sqrt5}{2}\right) + D\cdot\left(\tfrac{1-\sqrt5}{2}\right) \\ &f_2 = 1 \text{ so } 1 = C\cdot\left(\tfrac{1+\sqrt5}{2}\right)^2 + D\cdot\left(\tfrac{1-\sqrt5}{2}\right)^2 \end{aligned}}$

Solving (*), which is linear, we get $C = \frac{1}{\sqrt5}$, $D = \frac{-1}{\sqrt5}$.