OH Dec 2

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(i) The set $\mathbb{R}^{\mathbb{R}}$ is uncountable and $\nexists f: \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ bijection.

(ii) X uncountable means $\nexists g: X \to \mathbb{N}$ bijection.

(iii) $X \times Y$: $X = \mathbb{R}$, $Y = \mathbb{R}^{\mathbb{R}}$ so $X \times Y = \mathbb{R} \times \mathbb{R}^{\mathbb{R}}$

uncountable

$|X| < |X \times Y|$

Problem 3:

* f surj. if $\forall y\ \exists x$ s.t. $y = f(x)$
* f not surj. if $\exists y$ s.t. $y \neq f(x)\ \forall x$

* f inj. if $f(x) = f(y) \Rightarrow x = y$,
* f not inj. if $\exists x \neq y$ s.t. $f(x) = f(y)$.

(a) $f: \mathbb{N} \longrightarrow \mathbb{N}$, $f(n) = 4n + 6$

inj.? $\left(f(n) = f(m) \overset{?}{\Rightarrow} n = m\right)$ $\left.\begin{array}{l} f(n) = 4n+6 \\ f(m) = 4m+6 \end{array}\right\}$ $f(n) = f(m)$ is $4n+6 = 4m+6$, simplifies to $n = m$. inj. ✓

surj.? $\left(\begin{array}{c}\text{Need to solve } m = f(n) \\ \text{for any } m.\end{array}\right)$ m given! need find n! $m = 4n+6$? $\Leftrightarrow$ $(m-6)\cdot\frac{1}{4} = n$.

might not have $n \notin \mathbb{N}$.

counter-ex. E.g. $m = 1$, then $1 = f(n)$ iff $4n+6 = 1$, i.e. $n = -5/4 \notin \mathbb{N}$. $\leadsto$ NOT surj.!

(b) $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(x) = -x^2 + 3x + 5$.

inj.? $f(x) = f(y) \Leftrightarrow -x^2+3x+5 = -y^2+3y+5 \Leftrightarrow x(3-x) = y(3-y)$

stuck, could happen $x = 0$, $y = 3$

Indeed, $f(0) = 5$, $f(3) = 5$ so f not inj.

surj.? For y need x s.t. $y = f(x)$. Choose $y = 0$, then

$f(x) = y \Leftrightarrow f(x) = 0 \Leftrightarrow -x^2+3x+5 = 0 \Rightarrow x \notin \mathbb{Z}$.

surj.? for y need x s.t. $y = f(x)$ ...

$f(x)=y \Leftrightarrow f(x)=0 \Leftrightarrow -x^2+3x+5=0 \Rightarrow x \notin \mathbb{Z}$.

So NOT surj. b/c $\nexists x \in \mathbb{Z}$ s.t. $f(x)=0$.

(d) $f: \mathbb{Q} \setminus \{0\} \longrightarrow \mathbb{Q} \setminus \{0\}$, $f(x) = \frac{1}{x^2}$.

inj.? $f(x) = f(y) \Leftrightarrow \frac{1}{x^2} = \frac{1}{y^2}$ but could be $x = -y$

Ex: $\left.\begin{array}{l} f(1) = 1 \\ f(-1) = 1 \end{array}\right\}$ f not inj.! ✓

surj.? $\underbrace{y}_{\text{given}} = \underbrace{f(x)}_{\text{solving for } x}$? $\Leftrightarrow y = \frac{1}{x^2} \Leftrightarrow x = \frac{1}{\sqrt{y}} = \frac{1}{y}\sqrt{y}$ might not be $\mathbb{Q}$!

Ex: $y = \frac{1}{2}$ then $f(x) = \frac{1}{2} \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm\sqrt{2} \notin \mathbb{Q}$. not ~~surj~~!

Problem 4: Yoga = "find bijections to sets that we already the cardinality of"

(a) $X_1 = \{2n : n \in \mathbb{N}\}$ even numbers.

Sol^n 1: if A is countable and $X \subseteq A$, then X is countable.

Since $X_1 \subseteq \mathbb{N}$, and $\mathbb{N}$ is countable, X_1 countable.

Sol^n 2: Find a bijection to $\mathbb{N}$; e.g. $f: \mathbb{N} \longrightarrow X_1$, $n \longmapsto f(n) = 2n$.

This (prove this!) f is inj. and surjective to X_1. ∎

(b) $X_2 = \mathbb{Q} \times \mathbb{Q}$ is countable: since $\mathbb{Q} \underset{\text{bij.}}{\simeq} \mathbb{Z}$,

$|\mathbb{Q}| = |\mathbb{Z}|$

so $|\mathbb{Q} \times$ ☺$|$

(b) $X_2 = \mathbb{Q} \times \mathbb{Q}$ is countable: since $\mathbb{Q} \underset{bij.}{=} \mathbb{Z}$,

$$\text{card}(\mathbb{Q} \times \mathbb{Q}) = \text{card}(\mathbb{Z} \times \mathbb{Z}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{N}).$$

$\mathbb{Z}$ bij.

$\mathbb{Z} \cong \mathbb{Q}$ in class

in class with spiral!

in class done explicitly.

so $|\mathbb{Q} \times ☺|$ $=$ $|\mathbb{Z} \times ☺|$

(c) $X_3 = \{x \in \mathbb{R} : x > 3\}$? (uncountable!)

Sol^n 1: Adapt proof for $\mathbb{R}$ to this set X_3. For that show $\exists$ injection from $\underline{\underline{B}}$ binary seq. to $(4,5) \subseteq X_3$. $\Rightarrow$ uncountable $\blacksquare$

0101110 $\longrightarrow$ 4.0101110 ...

Sol^n 2: Consider $f : X_3 \longrightarrow \mathbb{R}_{>0}$, $f(x) = \dfrac{1}{x-3}$ bij.

So $\text{card}(X_3) = \text{card}(\mathbb{R}_{>0})$. Consider $g : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$ bij., $x \longmapsto \exp(x)$

then $\text{card}(\mathbb{R}) = \text{card}(\mathbb{R}_{>0})$. $\blacksquare$

Hence $\text{card}(X_3) = \text{card}(\mathbb{R})$, so uncountable.

(d) $X_4 = \{$ irrational numbers $\underline{\underline{\text{not}}}$ of the form $\sqrt[n]{2}$ $\} \subseteq \mathbb{I}$ irrationals

it'll be uncountable!

Sol^n: Remember if A and B are countable, then $A \cup B$ countable.

i.e. if A countable and $A \cup B$ uncountable, then B uncountable.

$$\mathbb{I} = X_4 \cup (\mathbb{I} \setminus X_4) = X_4 \cup \{\sqrt[n]{2}, \text{ for } n \in \mathbb{N}\}$$

$f : \mathbb{N} \to \{\sqrt[n]{2}\}$ bij, $n \mapsto f(n) = \sqrt[n]{2}$

uncountable

?

n bij.

countable b/c indexed by $\mathbb{N}$.

$\Downarrow$

X_4 uncountable.

Prob. 2. $\sqrt[5]{3}$ irrational

we can do that by ... num. & den. p/q

Prob. 2. $\sqrt[5]{3}$ irrational

Sol^{n}:

we can do that by simplif. num. & den. p/q

By contr., $\exists p,q \in \mathbb{Z}$ s.t. $\sqrt[5]{3} = p/q$, $\gcd(p,q)=1$.

Contradiction!

then $3q^5 = p^5$. Now $3 \mid 3q^5$ (lhs), so $3 \mid p^5$.

Since $3 \mid p^5$, then $3 \mid p$, so $3^5 \mid p^5$. Hence

$\gcd(q,p) \neq 1$ (at least 3)

$3^5 \mid 3 \cdot q^5$ so $3^4 \mid q^5$, so $3 \mid q^5$ so $3 \mid q$