OH Dec 3

Thursday, December 3, 2020 8:58 AM

Prob. 2.(b): $\sqrt{p} \notin \mathbb{Q}$ if p prime. — $p \mid a \cdot b \Rightarrow p \mid a$ or $p \mid b$

in particular, $p \mid a^n$ then $p \mid a$.

Sol^n: By contrad., suppose $\boxed{\sqrt{p} = \frac{a}{b}}$ (*) with $a, b \in \mathbb{Z}$ and wlog that $\gcd(a,b) = 1$.

(*) $p \cdot b^2 = a^2$, so $p \mid p \cdot b^2$ and thus $p \mid a^2$, $p \mid a$.

If $p \mid a$ then $p^2 \mid a^2$, so $p^2 \mid p \cdot b^2$, so $p \mid b^2$ so $p \mid b$.

Hence $p \mid \gcd(a,b)$, so $\gcd(a,b) \neq 1$, a contradiction ∎

Prob. 5. (a) This is equiv. to $\boxed{\text{"}X \text{ countable}, Y \text{ countable} \Rightarrow X \times Y \text{ countable"}}$ (#)

$\operatorname{card}(X) \leq \operatorname{card}(\mathbb{N})$

Sol^n: We prove (#) since $\operatorname{card}(X) \leq \operatorname{card}(\mathbb{Z})$, $\operatorname{card}(Y) \leq \operatorname{card}(\mathbb{Z})$.

$\Rightarrow \operatorname{card}(X \times Y) \leq \operatorname{card}(\mathbb{Z} \times \mathbb{Z}) = \operatorname{card}(\mathbb{Z})$ so countable ∎

done in lecture

(b) $X = [0,1] \times [0,1]$ uncountable. ~ $[0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

Sol^n: Since
$$X \longrightarrow [0,1], \quad (a,b) \mapsto a$$
is a surjection, $\Rightarrow \operatorname{card}(X) \geq \operatorname{card}([0,1])$.

Now, the injection $i : B \hookrightarrow \mathbb{R}$ actually lands inside $[0,1]$.

binary seq.

Hence $i : B \hookrightarrow [0,1]$ is an injection, so $\boxed{\operatorname{card}(B) \leq \operatorname{card}([0,1])}$.

In conclusion:

proven in class

$$\operatorname{card}(X) \geq \operatorname{card}([0,1]) \geq \operatorname{card}(B) \not\leq \operatorname{card}(\mathbb{N});$$

thus $\operatorname{card}(X) \not\leq \operatorname{card}(\mathbb{N})$, so X uncountable ∎

card $\lneq$	card $\lneq$	card
finite sets	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{Z}\times\mathbb{Z}$	$\mathbb{R}, \mathbb{I}, [0,1]$

countable

finite countable

infinite countable

$X \hookrightarrow Y$ injection, then $card(X) \leq card(Y)$

$X \twoheadrightarrow Y$ surjection, then $card(X) \geq card(Y)$

Prob. 4. (d): $X_4 = \{ x \in \mathbb{I} : x \neq \sqrt[n]{2} \}$

<u>Solⁿ:</u> $\underbrace{\mathbb{I}}_{\text{unc.}} = X_4 \cup \underbrace{\{ \sqrt[n]{2} \}}_{\text{countable}} \Rightarrow X_4$ uncountable.

$\{\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \sqrt[6]{2}, \dots\}$

$\{1, 2, 3, 4, 5, 6, 7, \dots\}$

$\mathbb{N} \xrightarrow{(*)} \{ \sqrt[n]{2} \}$

$n \mapsto \sqrt[n]{2}$.

bijection

(using A count, B count $\Rightarrow A \cup B$ count.)

Prob. 3(e): $f(x) = 5x^3 - 9$, $f: \mathbb{R} \to \mathbb{R}$.

<u>Solⁿ:</u> inj? $f(x) = f(y) \Leftrightarrow 5x^3 - 9 = 5y^3 - 9 \Leftrightarrow x^3 = y^3$

$\iff x = y$, so injective.

you can just use that

surj.? $y = f(x) \iff y = 5x^3 - 9 \iff x^3 = \frac{1}{5}(y+9)$

given find x

$\iff x = \sqrt[3]{\frac{1}{5}(y+9)} \in \mathbb{R}$ $\blacksquare$