

Prob. 2.(b):  $\sqrt{p} \notin \mathbb{Q}$  if  $p$  prime.  $p|a \cdot b \Rightarrow p|a$  or  $p|b$   
 in particular,  $p|a^2$  then  $p|a$ .

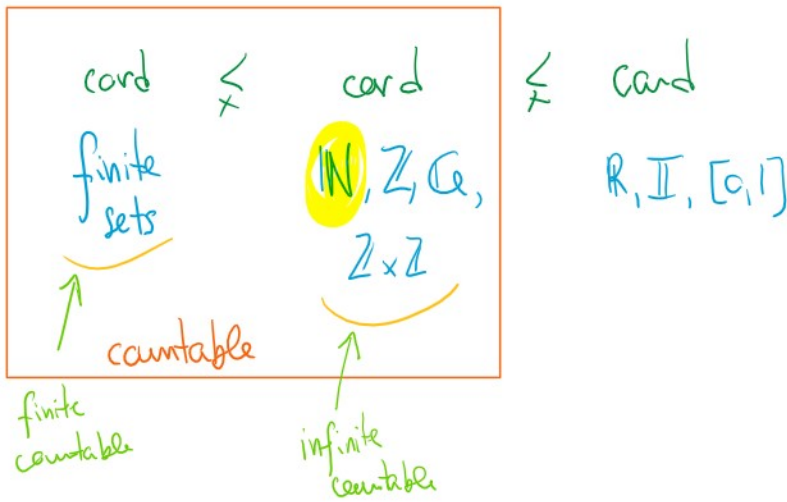
Sol<sup>n</sup>: By contrad., suppose  $\sqrt{p} = \frac{a}{b}$  with  $a, b \in \mathbb{Z}$  and wlog that  $\gcd(a, b) = 1$ .  
 (\*)  $p \cdot b^2 = a^2$ , so  $p|p \cdot b^2$  and thus  $p|a^2$ ,  $p|a$ .  
 If  $p|a$  then  $p^2|a^2$ , so  $p^2|p \cdot b^2$ , so  $p|b^2$  so  $p|b$ .  
 Hence  $p|\gcd(a, b)$ , so  $\gcd(a, b) \neq 1$ , a contradiction.  $\square$

Prob. 5. (a) This is equiv. to " $X$  countable,  $Y$  countable  $\Rightarrow X \times Y$  countable." (#)  
 $\text{card}(X) \leq \text{card}(\mathbb{N})$

Sol<sup>n</sup>: We prove (#) since  $\text{card}(X) \leq \text{card}(\mathbb{Z})$ ,  $\text{card}(Y) \leq \text{card}(\mathbb{Z})$ ,  
 $\Rightarrow \text{card}(X \times Y) \leq \text{card}(\mathbb{Z} \times \mathbb{Z}) = \text{card}(\mathbb{Z})$  so countable.  $\square$   
 done in lecture

(b)  $X = [0, 1] \times [0, 1]$  uncountable.  $\sim [0, 1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ .

Sol<sup>n</sup>: Since  $X \rightarrow [0, 1]$  is a surjection  $\Rightarrow \text{card}(X) \geq \text{card}([0, 1])$ .  
 $(a, b) \mapsto a$   
 Now, the injection  $i: \mathbb{B} \hookrightarrow \mathbb{R}$  actually lands inside  $[0, 1]$ .  
 Hence  $i: \mathbb{B} \hookrightarrow [0, 1]$  is an injection, so  $\text{card}(\mathbb{B}) \leq \text{card}([0, 1])$ .  
 In conclusion:  
 $\text{card}(X) \geq \text{card}([0, 1]) \geq \text{card}(\mathbb{B}) \geq \text{card}(\mathbb{N})$ ;  
 thus  $\text{card}(X) \geq \text{card}(\mathbb{N})$ , so  $X$  uncountable.  $\square$



$X \hookrightarrow Y$  injection, then  $\text{card}(X) \leq \text{card}(Y)$   
 $X \twoheadrightarrow Y$  surjection, then  $\text{card}(X) \geq \text{card}(Y)$

Prob. 4(d):  $X_4 = \{x \in \mathbb{I} : x \neq \sqrt[n]{2}\}$

Sol<sup>n</sup>:  $\mathbb{I} = X_4 \cup \{\sqrt[n]{2}\}$   $\Rightarrow X_4$  uncountable.

unc.      countable

$\{\sqrt[2]{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \sqrt[6]{2}, \dots\}$   
 $\{1, 2, 3, 4, 5, 6, 7, \dots\}$

$\mathbb{N} \xrightarrow{(*)} \{\sqrt[n]{2}\}$   
 $n \mapsto \sqrt[n]{2}$   
 bijection

using A count, B count  $\Rightarrow A \cup B$  count.

Prob. 3(e):  $f(x) = 5x^3 - 9$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

Sol<sup>n</sup>: inj?  $f(x) = f(y) \Leftrightarrow 5x^3 - 9 = 5y^3 - 9 \Leftrightarrow x^3 = y^3$

$\Leftrightarrow x = y$ , so injective.  
you can just use that

surj. ?

$$y = f(x) \Leftrightarrow y = 5x^3 - 9 \Leftrightarrow x^3 = \frac{1}{5}(y+9)$$

given  $y$  find  $x$

$$\Leftrightarrow x = \sqrt[3]{\frac{1}{5}(y+9)} \in \mathbb{R} \quad \square$$