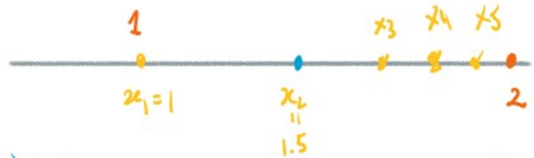
OH Mon Nov 16

Monday, November 16, 2020 4:06 PM

- Prob 4 PSet 5
- Density of $\mathbb{Q}$ in $\mathbb{R}$ & $\sqrt{2} \notin \mathbb{Q}$
- Prob 3. ✓

Prob. 4: $\boxed{x_{n+1} = x_n/2 + 1}$, $x_1 = 1$ (*)



$x_1 = 1$
$x_2 = 1.5$
$x_3 = x_2/2 + 1 = 1.75$
$x_4 = 1.875$
$x_5 = 1.9375$

Sol^n: We'll prove conv. via MCT (increasing & bound. above)

(i) Bounded above: we guess $x_n \leq 2$.

$$\boxed{\underbrace{x_{n+1} \leq 2}_{\text{we want}} \overset{(*)}{\iff} x_n/2 + 1 \leq 2 \iff x_n + 2 \leq 4 \iff \underbrace{x_n \leq 2}_{\text{assume}}.}$$

By induction, since $\underbrace{x_1 \leq 2}_{\text{base case}}$ we get $x_n \leq 2 \ \forall n \in \mathbb{N}$.

(ii) Increasing: we want $x_{n+1} \geq x_n$.

$$\boxed{x_{n+1} \geq x_n \iff x_n/2 + 1 \geq x_n \iff 2 \geq x_n} \text{ true by (i)}$$

☒

Prob. 3: $y_n = \sum_{k=1}^{n} k^{-2} = \frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2}}$

$y_1 = 1 = 1$,
$y_2 = 1 + 1/4 = 1.25$,
$y_3 = 1 + 1/4 + 1/9 = 1.3\hat{61}$

$y_4 = 1 + 1/4 + 1/9 + 1/16 \simeq 1.42$
$y_5 = 1 + 1/4 + 1/9 + 1/16 + 1/25 \simeq 1.46$

~ 1.64

Sol^n: (i) Increasing: since $\frac{1}{n^2} \geq 0$, $\boxed{y_n = y_{n-1} + \underbrace{\frac{1}{n^2}}_{\geq 0} \geq y_{n-1}.}$

(ii) Bounded above: we guess $y_n \leq 2$?

$y_{n+1} \leq 2 \iff y_n + \frac{1}{(n+1)^2} \leq 2 \iff y_n \cdot (n+1)^2 \leq 2(n+1)^2 - 1$

$\iff$? $y_n \cdot n^2 + 2n y_n + y_n \leq 2n^2 + 4n + \underbrace{(2-1)}_{=3}$

directly, we get stuck!

We know how to sum series with geometric radius

$$\sum_{k=1}^{N} r^k = \frac{1 - r^{N+1}}{1 - r} \quad \text{if } r < 1$$

we bound $\sum_{k=1}^{n} \frac{1}{k^2}$ above by a geometric series (we'll do $r = 1/2$)

For that,

$$(*) \quad \frac{1}{1^2} + \overbrace{\frac{1}{2^2} + \frac{1}{3^2}}^{2^1} + \overbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}^{2^2} + \frac{1}{8^2} + \frac{1}{9^2}$$

$$\frac{1}{1^2} \leq 2^0, \quad \frac{1}{2^2} \leq \frac{1}{2^2}, \quad \frac{1}{3^2} \leq \frac{1}{2^2}, \quad \frac{1}{4^2} \leq \frac{1}{(2^2)^2}, \quad \frac{1}{5^2} \leq \frac{1}{(2^2)^2}, \quad \frac{1}{6^2} \leq \frac{1}{4^2}$$

when we hit $\frac{1}{2^k}$, $\quad \frac{2^1}{4^1} \quad \frac{2^2}{4^2}$

then we bound any $\frac{1}{m^2}$ as follows: $2^k \leq m \leq 2^{k+1} \rightsquigarrow (2^k)^2 \leq m^2 \leq (2^{k+1})^2$

if

$$\rightsquigarrow \frac{1}{(2^{k+1})^2} \leq \frac{1}{m^2} \leq \frac{1}{(2^k)^2}$$

use this!

The sum (*) is now bounded above by

$$y_n \leq \sum_{k=0}^{n} 2^k \cdot \frac{1}{4^k},$$

$r = \frac{1}{2}$

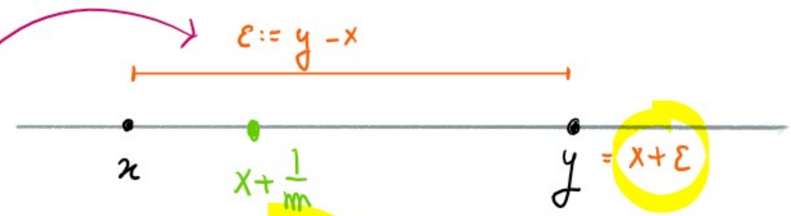
then in the limit $\sum_{k=0}^{\infty} 2^k/4^k = \sum_{k \geq 0} \frac{1}{2^k} = 2.$ ∎

Review density: $\boxed{\forall x, y \in \mathbb{R},\ x \leq y,\ \exists r \in \mathbb{Q} \text{ s.t. } x < r < y}$.

Sol^n: we need $n, m \in \mathbb{Z}$

s.t. $x < \frac{n}{m} < y$ ..

our $r \in \mathbb{Q}$



(Step 1) By Prop. 10.4., $\forall \varepsilon > 0\ \exists m \in \mathbb{N}$ s.t. $\frac{1}{m} < \varepsilon$. Thus

$$\boxed{x < x + \frac{1}{m} < x + \varepsilon = y}$$

$\frac{xm+1}{m}$ → not necessarily in $\mathbb{Z}$, so what they not necer. in $\mathbb{Q}$

To make $xm+1$ related to an integer we consider

$n \in \mathbb{N}$ such that $x \cdot m \leq n$. (choose n minimal : $\boxed{n-1 \leq x \cdot m \leq n}$) (*) Thm. 10.1

(*) $\frac{n-1}{m} \leq x \leq \frac{n}{m}$ ⇝ good, $x \leq \frac{n}{m}$, $\frac{n}{m} \in \mathbb{Q}$,

if we can prove $\frac{n}{m} < y$, done!

$\boxed{\text{Is it true that } \frac{n}{m} < y \text{ ?}}$

$$\boxed{\frac{n}{m} \leq x + \frac{1}{m} \overset{\frac{1}{m} < \varepsilon}{<} x + \varepsilon = y.}$$

remember (*) $n - 1 \leq mx \iff n \leq mx + 1 \Rightarrow \frac{n}{m} \leq x + \frac{1}{m}$ ▨