OHNov18
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• PSet5 : 2cd, 5ac, 3b, 6cbd

Prob. 2. (c) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Sol^n: $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t. $\left| \frac{n!}{n^n} - \overset{\text{limit}}{0} \right| < \varepsilon$ if $n \geq n_0$.

i.e. show that $\forall \varepsilon > 0$ we have

$$\frac{n!}{n^n} < \varepsilon \quad \text{for } n \text{ large enough.}$$

The key is bounding above $n!/n^n$ by something we know is small (it'll be $1/n$)

$\boxed{\frac{n!}{n^n} \overset{?}{\leq} \frac{1}{n}}$: b/c $\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdot \ldots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \leq \frac{1}{n}$ ✓

By Prop 10.4, $\exists n_0 \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$ for $\forall n \geq n_0$. Hence, for this same n_0

we have $\left| \frac{n!}{n^n} - 0 \right| \leq \left| \frac{1}{n} \right| < \varepsilon$ for $\forall n \geq n_0$. ☒

(also, if you want, Prop 10.4 $\exists n_0$ s.t. $\frac{1}{n_0} < \varepsilon$, then $\forall n \geq n_0$ satisfies $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$.)

(d) $\lim_{n\to\infty} \frac{3n^2-1}{n^2+n} = 3$.

Soln: we need that $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t.

$$\left| \frac{3n^2-1}{n^2+n} - 3 \right| < \varepsilon \quad \text{if } n \geq n_0.$$

How to get that?

$\circledast$ $\left| \frac{3n^2-1}{n^2+n} - 3 \right| \overset{?}{<} \frac{3}{n}$ $:\Leftrightarrow$ $\left| \frac{3n^2-1-3n^2-3n}{n^2+n} \right| < \frac{3}{n}$ $\Leftrightarrow$

(*) $\boxed{\left| \dfrac{3n^2-1}{n^2+n} - 3 \right| < \dfrac{3}{n}}$ $\iff$ $\left| \dfrac{}{n^2+n} \right|$ n

$\iff \left| \dfrac{3n+1}{n^2+n} \right| \leq \dfrac{3}{n}$. $\leftarrow$ $\left| \dfrac{3n+1}{n^2+n} \right| \leq \left| \dfrac{3n+3}{n^2+n} \right| = \dfrac{3}{n}$

By Prop. 10.4. $\forall \varepsilon_0 > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\dfrac{1}{n_0} < \varepsilon_0$. Now given our initial $\varepsilon > 0$, apply Prop. 10.4 to $\varepsilon_0 = \dfrac{\varepsilon}{3}$; then we get $n_0 \in \mathbb{N}$ s.t. $\dfrac{1}{n_0} \leq \varepsilon_0 = \dfrac{\varepsilon}{3}$,

that means $\boxed{\dfrac{3}{n_0} \leq \varepsilon}$.

By (*) $\boxed{\left| \dfrac{3n^2-1}{n^2+n} - 3 \right| < \dfrac{3}{n}}$, we get $\left| \dfrac{3n^2-1}{n^2+n} - 3 \right| \leq \dfrac{3}{n} < \varepsilon$ if $n \geq n_0$. ∎

Prob. 3.(b). $y_n = \sum_{k=1}^{n} \dfrac{1}{k^2}$, increasing ✓, bounded above?

we know: $\sum_{k \geq 0} r^k < \infty$ if $r < 1$. $\leftarrow$ geom. series!

y_{21} is $\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2} = \frac{4}{4^2}$

$$\boxed{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{12^2}+\frac{1}{13^2}+\frac{1}{14^2}+\frac{1}{15^2}+\frac{1}{16^2}+\frac{1}{17^2}+\frac{1}{18^2}+\frac{1}{19^2}+\frac{1}{20^2}+\frac{1}{21^2}}$$

$\wedge$ 1

each term bounded by $\frac{1}{2^2}$ $\quad \mathbin{/\!\wedge}$ $\frac{2}{2^2} = \frac{1}{2}$

each term bounded by $\frac{1}{4^2}$ $\quad \mathbin{/\!\wedge}$ $\frac{4}{4^2} = \frac{1}{4} = \frac{1}{2^2}$

8 terms, each term bounded by $\frac{1}{8^2}$ $\quad \mathbin{/\!\wedge}$ $\frac{8}{8^2} = \frac{1}{8} = \frac{1}{2^3}$

each term bounded by $1/16^2$ $\quad \mathbin{/\!\wedge}$ $\frac{16}{16^2} = \frac{1}{16} = \frac{1}{2^4}$

until $\frac{1}{32^2}$

$y_n \leq \sum^{n} \dfrac{1}{2^k} = 2$. $\left\{ y_n = \sum^{n} \dfrac{1}{k^2},\ \text{for each } k \text{ we have } \exists! m \text{ s.t.} \right.$

$y_n \leq \underbrace{\sum_{k\geq 0} \frac{1}{2^k}}_{} = \underset{\substack{\uparrow \\ \frac{1}{1-r},\ r=\frac{1}{2}}}{2}$, $\left\{ y_n = \sum_{k\geq 1}^{n} \frac{1}{k^2}\right.$, for each k we have $\exists! m$ s.t. $2^m \leq k \leq 2^{m+1}$, then $\left.\frac{1}{k} \leq 2^{-m}\right\}$

Prob. 5. (a). Suppose (x_n) is convergent, then $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$

we have $|x_n - L| < \varepsilon$ for $\underbrace{n >> 1}_{n \text{ large enough}}$. $\exists n_0$ s.t.

this is true $\forall \varepsilon > 0$, it must be true for $\varepsilon = 1$.

Then we have $|x_n - L| \leq 1$ for $n \geq n_0$

$\Leftrightarrow$ $\boxed{|x_n| \leq |L| + 1}$ for $n \geq n_0$ — we have bounded the terms $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$

but not $x_1, x_2, \ldots, x_{n_0-1}$.

Choose $M := \max\{\overbrace{x_1, x_2, \ldots, x_{n_0-1}}^{\text{finitely many!}}, |L|+1\}$, then this

bounds x_n $\forall n \in \mathbb{N}$, i.e. — takes care of $n \geq n_0$

$$|x_n| \leq M.$$

So (x_n) is bounded. ▨

Prob. 5. (c). $\exists$ seq. (x_n) bounded but no convergent.

we give an example that this indeed possible:

$$x_n = (-1)^n.$$

Then $|x_n| = |(-1)^n| = 1 \leq 2$ — 2 is upper bounded

Then $|X_n| = |(-1)^n| = 1 \leq 2$.

$\boxed{\text{Now need to show } X_n \text{ is not convergent.}}$

Prob. 6. (b) $x_n = \frac{(-1)^n}{n}$ is convergent. ⇝ true and proven by ε-def, $\lim_{n \to \infty} x_n = 0$

(c) (X_n) is s.t. $|X_n|$ converges, then X_n conv.? use Prop. 10.4!

$x_n = (-1)^n$ does not converge, but $y_n = |x_n| = 1$ so convergent. → Counterexample $\exists$, so false!

(d) $(X_n), (y_n)$ unbounded, then $(z_n) = (X_n \cdot y_n)$ is unbounded.

$\exists$ counterexample: $x_n = (\underset{x_1}{0}, \underset{x_2}{1}, \underset{x_3}{0}, \underset{x_4}{2}, \underset{x_5}{0}, \underset{x_6}{3}, \underset{x_7}{0}, \underset{x_8}{4}, \underset{x_9}{0}, \underset{x_{10}}{5}, \ldots)$

i.e. $X_{2k+1} = 0$ and $X_{2k} = k$. This is UNBOUNDED!

$y_n = (\underset{x_1}{1}, \underset{x_2}{0}, \underset{x_3}{2}, \underset{x_4}{0}, \underset{x_5}{3}, \underset{x_6}{0}, \underset{x_7}{4}, \underset{x_8}{0}, \underset{x_9}{5}, 0, 6, 0, \ldots)$

i.e. $y_{2k+1} = k+1$, $y_{2k} = 0$. This is also unbounded.

$(z_n) = (X_n \cdot y_n)$ is the sequence $z_n = 0 \ \forall n \in \mathbb{N}$.

In particular, z_n bounded!