OH Mon Nov 9 · Q6 (PSet4) · Q2 c&d · Q3

Monday, November 9, 2020 4:04 PM

Prob. 2 (c) $C = (0, \infty)$

Sol^n: We guess that $\inf(C) = 0$, and $\nexists \sup(C)$ i.e. C is unbounded above.

we use: if $A \subseteq B$ & B bounded above then A bounded above

(1) $\nexists \sup(C)$. : Since $\mathbb{N} \subseteq C$, and $\mathbb{N}$ is not bounded above, then C is not bounded above.

"if $M \in \mathbb{R}$ was upper bound then $M > 0$," $M+1 > 0$ so $M+1 \in C$ so M cannot be upper bound. } different argument 2

(2) $\inf(C) = 0$? First, 0 is a lower bound because if $x \in C$, then $0 < x$. so $0 < x, \forall x \in C$.

Now, suppose $\inf(C) \neq 0$, then $\inf(C) > 0$ so $\frac{\inf(C)}{2} > 0$.

But $\frac{\inf(C)}{2} \in C$ but $\underbrace{\frac{\inf(C)}{2}}_{\in C} < \inf(C)$, which is a contradiction! ☒

Prob. 3 : Find $\inf(A)$ and $\sup(A)$, with $A = \{3 - \frac{1}{n} : n \in \mathbb{N}\} = \{\underbrace{3 - \frac{1}{1}}_{= 2}, 2.5, 2.\hat{6}, 2.75, 2.8, \dots, 2.9\dots9, \dots\}$

Sol^n: First, $\inf(A) = 2$.

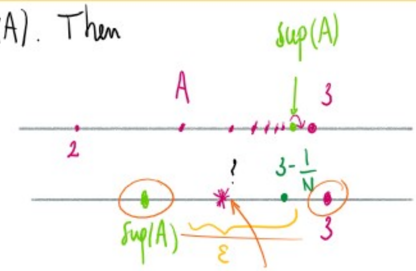
↳ (i) 2 is a lower bound : $3 - \frac{1}{n} \overset{?}{\geq} 2$, for $n = 1$ we have $3 - \frac{1}{n} = 2$. ← base case

in general $3 - \frac{1}{n+1} > 3 - \frac{1}{n} \geq 2$. ← induction step

$\Rightarrow 2 \leq 3 - 1/n \; \forall n \in \mathbb{N}$.

↳ (ii) Since $2 \in A$, it must be the greatest lower bound.

$\boxed{\sup(A) = 3}$: By contradiction, suppose $3 \neq \sup(A)$. Then $\sup(A) < 3$, we will argue that then $\sup(A)$ is not an upper bound. Indeed, let



$3 - \varepsilon = \sup \Longleftrightarrow$ $\boxed{\varepsilon = 3 - \sup(A)} > 0$, then

Prop. 10.4 tells us $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. $\Longleftrightarrow -\varepsilon < -\frac{1}{N}$

Hence $3 - \frac{1}{N} \in A$, because $N \in \mathbb{N}$.

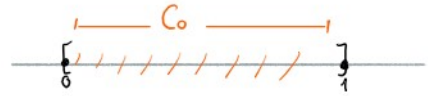
Also $\underbrace{3 - \frac{1}{N}}_{\in A} > 3 - \varepsilon = \sup(A)$

$\sup(A) < 3 - \frac{1}{N} \in A$ so $\sup(A)$ is NOT an upper bound ☒

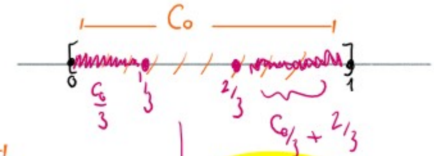
A

(Rmk: Had it been $3-\frac{1}{n^2}$, use Prop. 10.4 but also inequality $3-\frac{1}{n^2} \geq 3-\frac{1}{n} \overset{10.4}{\geq} 3-\varepsilon$.)

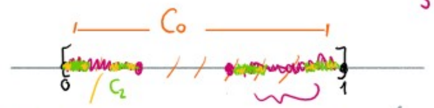
Prob. 6: $C_0=[0,1]$



C_1? $C_1 = \frac{C_0}{3} \cup \left(\frac{2}{3}+\frac{C_0}{3}\right)$



C_2?



$\boxed{C_1=[0,1/3]\cup[2/3,1]}$

C_2

$\boxed{[0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]}$

Prob. 5 $X=\left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}\right\}$ is bounded above.

Sol^n: Since, by the hint,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\frac{1}{k!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

Bin. thm

$k\leq n$

$n=1$

$\left(1+\frac{1}{1}\right)^1=2$

$$= 1+1+\left[\sum^{n}\frac{1}{k!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]$$

$n\geq 2$

first terms $n=1$

$n\geq 2$ $(n+1)$

$\frac{1}{k!}\leq\frac{1}{2^k}$

Lect. on Mond. Oct 12 "geometric sum"

$$\leq 1+1+\left[\sum_{k=4}^{n}\frac{1}{k!}\cdot 1\right] \leq 1+1+\left[\sum_{k=4}^{n}\frac{1}{2^k}\right] \leq 1+1+1=3 \quad \blacksquare$$

$\sum_{k=4}^{n}\frac{1}{2^k} \leq 1$

assume $2^k\leq k!$ $(k\geq 3)$

$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \xrightarrow{r=1/2}$

(equal to 1 when $\lim_{n\to\infty}$)

$n=2$ $n=3$ $n=4$

$n=2$ $\quad$ $n=3$ $\quad$ $n=4$

$$\frac{1}{2^1}+\frac{1}{2^2} < \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3} < \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$$

Also :

$$\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r} \overset{\uparrow \text{ for us } r=\frac{1}{2}}{=} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\cdot\Big(\underbrace{1-\left(\tfrac{1}{2}\right)^{n+1}}_{\overset{\wedge}{1/2}}\Big) \leq 1$$

check!

for any $n \in \mathbb{N}$