OH Mon Oct 19

Monday, October 19, 2020 4:01 PM

A problem similar to Prob. 3 by induction: Induction is on $n \in \mathbb{N}$, and for each $n \in \mathbb{N}$ prove statement for all $k \leq n$.

Example: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Sol$^n$: By induction; the base case is $n=0$: $\sum_{k=0}^{0} \binom{0}{k} = 2^0 \iff \binom{0}{0} = 2^0 \iff 1 = 1$. TRUE.

Induction step: assume $\underbrace{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$, want $\underbrace{\sum_{k=0}^{n+1} \binom{n+1}{k}}_{LHS} = 2^{n+1}$.

$\binom{n+1}{k} \overset{*}{=} \binom{n}{k} + \binom{n}{k-1}$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n+1} \binom{n}{k} + \binom{n}{k-1} = \sum_{k=0}^{n} \binom{n}{k} + \underbrace{\sum_{k=0}^{n+1} \binom{n}{k-1}}_{\overset{\text{reindex } l = k-1}{=} \sum_{l=0}^{n} \binom{n}{l}} \overset{\text{by induction}}{=} 2^n + 2^n = 2 \cdot 2^n = 2^{n+1} \checkmark.$$

$k = 0 \ldots k = n+1$

* $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} + \binom{n}{n+1}$ ; $0$

* $\binom{n}{-1} + \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$ ; $0$

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Prob. 3. (a) Once base case $n=0$ is done,

You assume $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ $\forall k \leq n+1$, you want $\binom{n+2}{k} = \binom{n+1}{k} + \binom{n+1}{k-1}$ $\forall k \leq n+2$

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Prob. 5. (Parenthesis) $P_n := \#$ ways of correctly write n parenthesis:

$P_0 = 1$

$P_1 = \#\{()\} = 1$ $()$, $\cancel{)(}$ 

1 2 3 4
(( ))

$P_2 = \#\{()(), (())\} = 2$, $\cancel{))((}$ less eq. than $4! = 24$

(( )) + (( ))

$P_3 = \#\{((())), (()()), ()()(), ()(()), (())()\} = 5$

$P_3 = \# \{ ((())), (()()), ()()(), ()(()), (())() \} = 5$

also for $P_4, P_5$, for $P_4 = 14$, $P_5 = 42$, $P_6 = 132$

Show that $\boxed{P_{n+1} = \sum_{k=0}^{n} P_k \cdot P_{n-k}}$

Hint of Sol$^n$: $1^{st}$ method: Compute $P_n$ directly, $P_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. you can show hard

$2^{nd}$ method: By recursion: $P_3 = \underbrace{P_0 \cdot P_2}_{2} + \underbrace{P_1 \cdot P_1}_{1} + \underbrace{P_2 \cdot P_0}_{2} = 2 + 1 \cdot 1 + 2 = 5$ ✓

$(a)(b)(c)$

$(a(b))(c)$

$P_2 = P_0 \cdot P_1 + P_1 \cdot P_0 = 1 + 1 = 2$

$()()$

$(())$

Think of $a \cdot (b \cdot c)$ or $(a \cdot b) \cdot c \rightsquigarrow (a \cdot b) \cdot (c \cdot d)$

out

$a \cdot (b \cdot (c \cdot d))$

out

In general, you have a product

n-1 dots

$(a_1 \cdot a_2 \cdot) \cdots \cdot (a_{n-1} \cdot a_n)$

at the left n-k numbers   out   at the right k numbers

$\rightsquigarrow$ depending on the k numbers left on the right, we parenthesize each side

$$\sum_{k=0}^{n-1} \underbrace{P_{n-k-1}}_{\text{on the left of the last}\,\cdot\,\text{out}} \cdot \underbrace{P_k}_{\text{on the right of the last}\,\cdot\,\text{out}} = P_n$$

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Prob. 1: 2n players, 6 player e.g.

$a_n = \#$ ways to pair up in first round w/ 2n players

a, b, c, d: $(ab) \leftrightarrow (cd)$, $(ac) \leftrightarrow (bd)$, $(ad) \leftrightarrow (bc)$

$a_1 = 1$

4 players $\sim$ $a_2 = 3$ $\times 3$

$a_3 = 15$ $\times 5$

$a_4 = 105$ $\times 7$

$a_5 = 945$ $\times 9$

6 players: $\underbrace{a, b, c, d, e, f}_{5}$

$a_4 = 105 \swarrow \times 7$

$a_5 = 945 \swarrow \times 9$

In general: suppose we have $2n+2$ players, how many ways $= a_{n+1}$?

Express $a_{n+1}$ in terms of $a_n, a_{n-1}, \ldots, a_1$.

Start at $(2n+2)$ players and pair 2 of them. ⇝ how many ways are there?

then left w/ $2n$ players so $a_n$ pairings:

$$a_{n+1} = \, ? \cdot a_n = (2n+1) \cdot a_n$$

counts how many pairs (2 players) can we choose out of $2n+2$ } ⇝ $2n+1$ options

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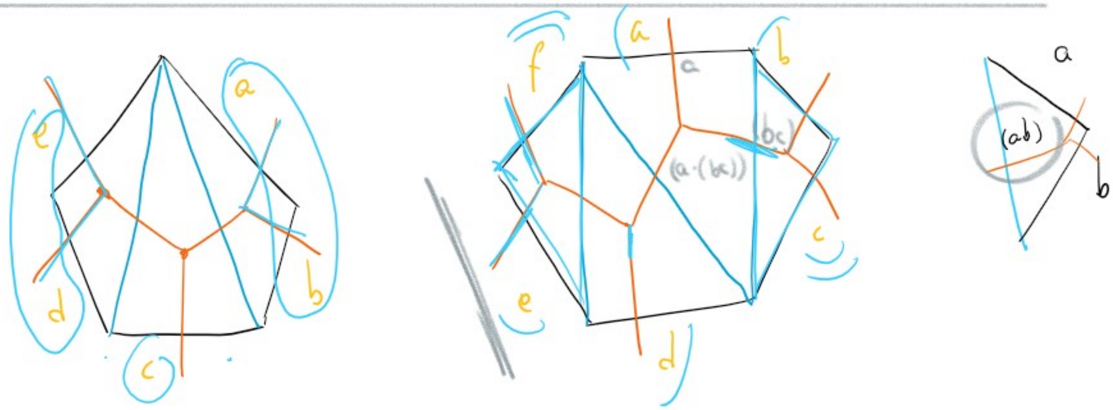


(1) add a different to all sides but one.
(2) parenthesize if ∃ triangle

} ⇝ read the parenthesized word at last side