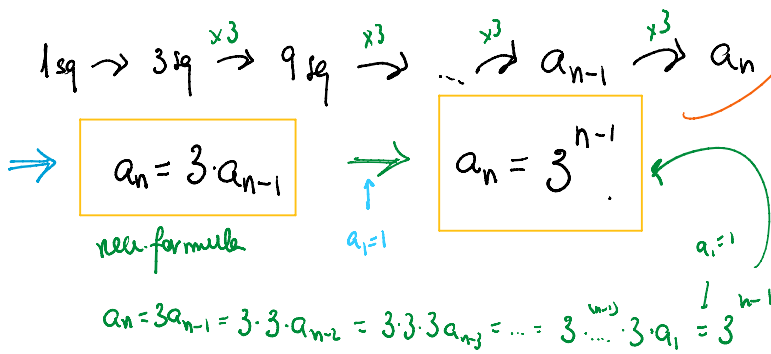


Recursion: given (a_n) a seq. $\xrightarrow{\textcircled{1}}$ express a_n in terms of a_1, a_2, \dots, a_{n-1}
 FINDING AND PROVING FORMULA
 $\xrightarrow{\textcircled{2}}$ closed formula \Rightarrow express a_n in terms of n (not a_{n-1}, \dots, a_1) $\xrightarrow{\text{induction!}}$

Prob. 2: (a) Recursion for a_n ?

\rightarrow (q1) How to express a_n in terms of a_{n-1}, \dots, a_1 ?

(q2) How to express a_n in terms of n ?



$a_n = 3^{n-1} \forall n.$
 (bc) $a_1 = 3^{1-1} = 3^0 = 1 \checkmark$

(ind/tp) assume

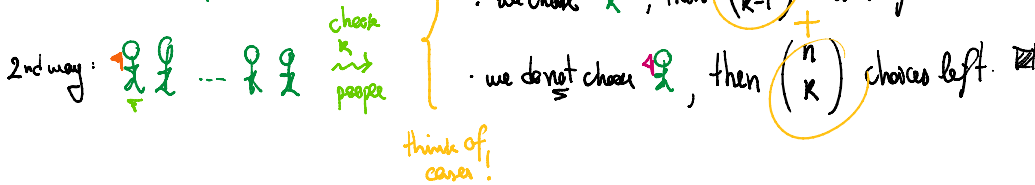
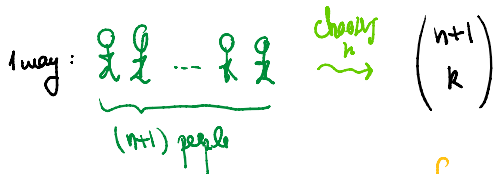
$a_n = 3^{n-1}$ want $a_{n+1} = 3^n$

we $a_{n+1} = 3 \cdot a_n = 3 \cdot 3^{n-1} = 3^n \checkmark$
 by ind

(b) this one features a geom. series, or you can multiply as in (a).

Prob 3: (a) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

1st method: combinatorial, count LHS in 2 different ways



2nd method: algebraic direct computation:

$\binom{n+1}{k} \stackrel{\text{definition}}{=} \frac{(n+1)!}{k!(n+1-k)!}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$$

definition

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

clean denominators and cancel factors. LHS = RHS.

3rd method: By induction, on $n \in \mathbb{N}$.

base case is $n=0$ and $k \leq n$ so $k=0$

$$\binom{0+1}{0} = \binom{0}{0} + \binom{0}{-1} \checkmark$$

We assume true for $n \in \mathbb{N}$ (and any k !)

We want true for $n+1$

only n goes up, k stays same

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{n+2}{k} = \binom{n+1}{k} + \binom{n+1}{k-1}$$

Start w/

$$\binom{n+2}{k} = \binom{n+1}{k} + \binom{n+1}{k-1} = \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-1} + \binom{n}{k-2} + \dots$$

+ more

Prob 4: it's all understanding the Binomial Thm:

$$(\#) (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(a) (a+b)^{11} = \binom{11}{0} a^{11} b^0 + \binom{11}{1} a^{10} b^1 + \binom{11}{2} a^9 b^2 + \dots + \binom{11}{6} a^5 b^6 + \dots$$

by (#) \Rightarrow coeff. of $a^5 b^6$ is $\binom{11}{6} = \frac{11!}{5!6!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5!} = \frac{11 \cdot 9 \cdot 8 \cdot 7}{12} = 11 \cdot 3 \cdot 2 \cdot 7 = 11 \cdot 42 = 462$

$$(b) \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = ?$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}$$

Why? If $x=1, y=-1$ in (#), the RHS is $\sum_{k=0}^n \binom{n}{k} (-1)^k$

Why? If $x=1, y=-1$ in (#), the RHS is $\sum_{k=0}^n \binom{n}{k} \cdot (-1)^k$

The LHS of (#) is $(1-1)^n = 0^n = 0$.

Since (#) says LHS = RHS, we conclude $\sum_{k=0}^n \binom{n}{k} \cdot (-1)^k = 0$.

(c) Same as (b) but now do $x=1, y=1$. \rightarrow LHS: $(1+1)^n = 2^n$.

Prob. 5:

$$x \cdot ((x \cdot x) \cdot (x \cdot x))$$

$$(x \cdot ((x \cdot x) \cdot x)) \cdot x$$

$$(x \cdot x) \cdot (x \cdot (x \cdot x))$$

$$x \cdot x \cdot x \cdot x \cdot x$$

In general

$$x \cdot x \cdot ((x \cdot x) \cdot x) \cdot \dots \cdot (x \cdot x) \cdot x \cdot x \cdot x$$

last happens at k

$n+1$ terms

parenthesized

P_k of doing this

parenthesized

P_{n-k} ways of doing this

$P_k \cdot P_{n-k}$ ways of parenthesizing

last prod. might happen at $n=0, 1, 2, \dots, n$

$$\sum_{k=0}^n P_k \cdot P_{n-k} = P_{n+1}$$

P3.(b) we have

$$\binom{n}{k} \stackrel{?}{=} \binom{n}{n-k}$$

we want

$$\binom{n+1}{k} \stackrel{???}{=} \binom{n+1}{n+1-k}$$

$$\binom{n+1}{k} = \frac{(n+1)}{(n+1-k)} \cdot \binom{n}{k} = \frac{(n+1)}{(n+1-k)} \cdot \binom{n}{n-k} \stackrel{\text{re-multiply}}{=} \binom{n+1}{n+1-k}$$

$$\binom{n+1}{k} = \frac{(n+1)!}{k! \cdot (n+1-k)!} = (n+1) \cdot \frac{n!}{k! \cdot (n-k)!} \cdot \frac{1}{(n+1-k)}$$

$$\binom{n+1}{k} = \frac{(n+1)!}{k! \cdot \underbrace{(n+1-k)!}} = (n+1) \cdot \frac{n!}{k! \cdot (n-k)!} \cdot \frac{1}{(n+1-k)}$$

$$\left. \begin{array}{l} P_0=1 \\ P_1=1 \\ P_2=2 \end{array} \right\} \begin{array}{l} P_3 = P_0 P_2 + P_1 P_1 + P_2 P_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5 \\ P_4 = P_0 P_3 + P_1 P_2 + P_2 P_1 + P_3 P_0 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14 \end{array}$$