OH Wed Oct 21

Wednesday, October 21, 2020 4:04 PM

• Recursion • P6 • P2 • P5 • P4.(b) ~ (d) • P3

Recursion: given (a_n) a seq. ① $\rightarrow$ express a_n in terms of $a_1, a_2, \dots, a_{n-1}$

FINDING AND PROVING FORMULA

② closed formula for a_n $\Rightarrow$ express a_n in terms of n (not $a_{n-1}, \dots, a_1$)

by why

induction!

Prob. 2: (a) Recursion for a_n?

$\rightsquigarrow$ (q1) How to express a_n in terms of $a_{n-1}, \dots, a_1$?

(q2) How to express a_n in terms of n?

$$1 \text{ sq} \rightarrow 3 \text{ sq} \xrightarrow{\times 3} 9 \text{ sq} \xrightarrow{\times 3} \dots \xrightarrow{\times 3} a_{n-1} \xrightarrow{\times 3} a_n$$

$$\Rightarrow \boxed{a_n = 3 \cdot a_{n-1}} \xrightarrow{a_1 = 1} \boxed{a_n = 3^{n-1}}.$$

rec. formula

$a_1 = 1$

$$a_n = 3a_{n-1} = 3 \cdot 3 \cdot a_{n-2} = 3 \cdot 3 \cdot 3 a_{n-3} = \dots = \underbrace{3 \cdot \dots \cdot 3}_{(n-1)} \cdot a_1 = 3^{n-1}$$

$a_n = 3^{n-1}$ Th.

(bc) $a_1 = 3^{1-1} = 3^0 = 1$ ✓

(ind step) assume $a_n = 3^{n-1}$ want $a_{n+1} = 3^n$.

we $a_{n+1} = 3 \cdot a_n = 3 \cdot 3^{n-1} = 3^n$ ✓

by ind

(b) this one features a geom. series, or you can multiply as in (a).

Prob 3: (a) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

$\boxed{1^{st} \text{ method}}$: Combinatorial, count LHS in 2 different ways

1 way: $\underbrace{\text{people}}_{(n+1) \text{ people}}$ $\xrightarrow{\text{choosing } k}$ $\binom{n+1}{k}$

2nd way: people $\xrightarrow{\text{choose } k}$ people

• we choose [person], then $\binom{n}{k-1}$ choices left.

+

• we do not choose [person], then $\binom{n}{k}$ choices left. ☒

think of cases!

$\boxed{2^{nd} \text{ method}}$: algebraic direct computation:

$$\binom{n+1}{k} := \frac{(n+1)!}{k!(n+1-k)!}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}.$$

definition

$$\binom{n+1}{k} \overset{\text{definition}}{:=} \frac{(n+1)!}{k!(n+1-k)!}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$$

$$\frac{(n+1)!}{k!(n+1-k)!} \overset{?}{=} \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \xrightarrow{\text{clean}} \text{LHS} = \text{RHS}.$$

denominators and cancel factors.

$\boxed{3^{rd} \text{ method}}$: $\boxed{\text{By induction, on } n \in \mathbb{N}.}$

base case is $n=0$: and $k \leq n$, so $k=0$ $\quad \binom{0+1}{0} = \binom{0}{0} + \cancel{\binom{0}{-1}}$ ✓. ($1 = 1 + 0$)

We assume true for $n \in \mathbb{N}$ (and any k!) $\rightsquigarrow$ we want true for $n+1$ — only n goes up, k stays same

$$\boxed{\binom{n+1}{k} \overset{\circledast}{=} \binom{n}{k} + \binom{n}{k-1}} \rightsquigarrow \binom{n+2}{k} = \binom{n+1}{k} + \binom{n+1}{k-1}$$

Start w/ $\binom{n+2}{k} = \underbrace{\binom{n+1}{k}}_{\text{use } \circledast} + \underbrace{\binom{n+1}{k-1}}_{\text{use } \circledast} = \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-1} + \binom{n}{k-2}$

+ more

Prob. 4: it's all understanding the Binomial Thm:

$$(\#) \quad \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} y^{k}}$$

(a) $(a+b)^{11} = \underset{k=0}{\binom{11}{0}} a^{11} b^{0} + \underset{k=1}{\binom{11}{1}} a^{10} \cdot b^{1} + \underset{k=2}{\binom{11}{2}} a^{9} \cdot b^{2} + \ldots + \underset{k=6}{\boxed{\binom{11}{6}}} \cdot a^{5} b^{6} + \ldots$

$n=1$

by (#) $\Rightarrow$ coeff. of $a^5 b^6$ is $\binom{11}{6} = \frac{11!}{5!\,6!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5!} = \frac{11 \cdot 9 \cdot 8 \cdot 7}{12} = 11 \cdot 3 \cdot 2 \cdot 7$

$= 11 \cdot 42 = 462$

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} \overset{?}{=} 0$

$$\sum_{n=0}^{n} (-1)^n \binom{n}{k}$$

Why? If $x=1$, $y=-1$ in (#), the RHS is $\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{k}$

why? If $x=1$, $y=-1$ in (#), the RHS is $\sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^k$

The LHS of (#) is $(1-1)^n = 0^n = 0$.

Since (#) says LHS = RHS, we conclude $\sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^k = 0$.

(c) Same as (b) but now do $x=1$, $y=1$. ⇝ LHS: $(1+1)^n = 2^n$.

Prob. 5:

$x\cdot((x\cdot x)\cdot(x\cdot x))$

$(x\cdot((x\cdot x)\cdot x))\cdot x$

$(x\cdot x)\cdot(x\cdot(x\cdot x))$

$x\cdot x\cdot x\cdot x\cdot x$

In general:

last · happens at k

n

n+1 terms

$$\underbrace{x\cdot x\cdot((x\cdot x)\cdot x)}_{\text{parenthesized}}\cdot\underbrace{\ldots\cdot(x\cdot x)\cdot x\cdot x\cdot x}_{\text{parenthesized}}$$

P_k of doing this

P_{n-k} ways of doing this

$P_k\cdot P_{n-k}$ ways of parenthesizing

$$\sum_{k=0}^{n} P_k\cdot P_{n-k} = P_{n+1}$$

last prod might happen at $k=0,1,2,\ldots,n$

P3.(b) we have

$$\boxed{\binom{n}{k} \overset{*}{=} \binom{n}{n-k}}$$

we want

$$\binom{n+1}{k} \overset{???}{=} \binom{n+1}{n+1-k}.$$

$$\binom{n+1}{k} = \frac{(n+1)}{(n+1-k)}\cdot\binom{n}{k} = \frac{(n+1)}{(n+1-k)}\cdot\binom{n}{n-k} \overset{\text{re multiply}}{=} \binom{n+1}{n+1-k}$$

$$\binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-k)!} = (n+1)\cdot\frac{n!}{k!\,(n-k)!}\,\frac{1}{(n+1-k)}$$

$$\binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-n)!} = (n+1)\cdot\frac{n!}{k!\,(n-k)!}\;\frac{1}{(n+1-k)}$$

$$\left.\begin{array}{l} P_0=1 \\ P_1=1 \\ P_2=2 \end{array}\right\} \quad \begin{array}{l} P_3 = P_0P_2 + P_1P_1 + P_2P_0 = 1\cdot2 + 1\cdot1 + 2\cdot1 = 5 \\ P_4 = P_0P_3 + P_1P_2 + P_2P_1 + P_3P_0 = 1\cdot5 + 1\cdot2 + 2\cdot1 + 5\cdot1 = 14. \end{array}$$