OH Mon Oct 26

Monday, October 26, 2020 2:10 PM

$\boxed{\text{Prob. 3 of Prac. Prob. II}}$ Last digit of 4^{100} ?

method 1: we want $4^{100} \equiv x \mod 10$, with $0 \leq x \leq 9$.

mod 10

$4^0 \equiv 1$ $\times 4$

$4^1 \equiv 4$ $\times 4$

$4^2 \equiv 6 \equiv -4$ $\times 4$

$4^3 \equiv 4$

$4^4 \equiv 6$

$4^5 \equiv 4$

6

4

6

4

$\Rightarrow$ powers of 4 have last digit 4 if exponent is odd and have last digit 6 if exp. is even.

$\Downarrow$ 100 even

$4^{100} \equiv 6 \mod 10$ ■

method 2: we want $4^{100} \equiv x \mod 2$ AND $4^{100} \equiv y \mod 5$ ($\Rightarrow$ will tell us value mod 10)

(*) 1st: $4^{100} \equiv 0 \mod 2$

2nd: $4^{100} \equiv ? \mod 5$ $\leftarrow$ $a^p \equiv a \mod p$ if p prime, in addition if $a \not\equiv 0 \mod p$ $a^{p-1} \equiv 1 \mod p$

||| also $(-1)^{100} \equiv 1 \mod 5$

$$\underbrace{\qquad\qquad\qquad\qquad}_{a^5 \equiv a,\ \text{also if } 5 \nmid a,\ a^4 \equiv 1.}$$

$a^5 \equiv a$

$a^4 \cdot \cancel{a} \equiv \cancel{a}$

$a^4 \equiv 1.$

$4^{100} \equiv (4^4)^{25} \equiv 1^{25} \equiv 1 \mod 5$

$\Rightarrow$ the only $\equiv 1 \mod 5$ between 1 and 10 are 1, 6 $\xRightarrow[\text{must be even}]{}$ $4^{100} \equiv 6 \mod 10$ ■

$\boxed{\text{Prob. 2 in Prac. Prob. II}}$: (a) Show $3 \nmid 4^{100}$.

Sol^n: Opt. 1 is directly $4^{100} = 2^{200}$, only 2's in prime dec., so done.

Opt. 2: work modulo 3 and show $4^{100} \not\equiv 0 \mod 3$.

Indeed, $4^{100} \equiv 1^{100} \equiv 1 \not\equiv 0 \mod 3$, so done. ■

$\boxed{\text{Prob. 5 Pract. Prob. I}}$: $\exists$ infinitely many primes of the form $6k+5$.

Sol^{n}: By contradiction, assume $\exists$ only FINITELY MANY primes of the form $6k+5$.

Call these $\{p_1, p_2, \dots, p_N\}$. Now consider the number

$$S := 6(p_1 \cdot p_2 \cdots p_N) - 1$$, which is of the form $6k+5$ by constr.

Now, $S \neq p_i$ because $S > p_i$. So S is not a prime of the form $6k+5$.

Since S is divisible by primes, we will reach a contradiction if:

contradiction $\exists$ ms

(1) $p_i \nmid S$, $1 \le i \le N$: why? if $p_i \mid S$ then $p_i \mid -1$ as $p_i \mid 6(p_1 \cdots p_N)$, so it cannot be.

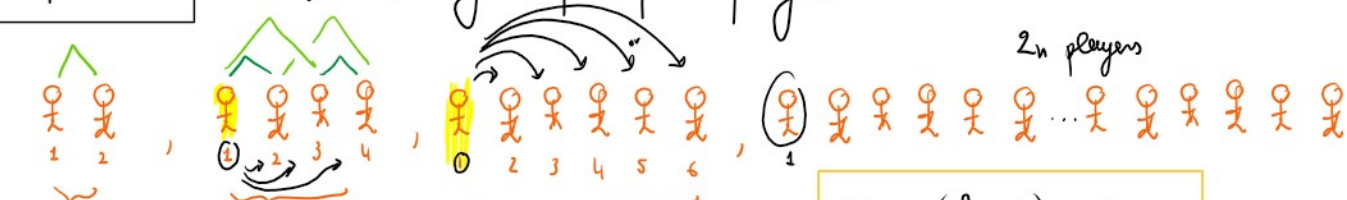
(2) S must be divided by p_i : why? S has a prime dec., primes must be $6k+1$ or $6k-1$.

If all divisors of S were $6k+1$, then S would be $6k+1$, but S is $6k-1$,

so $\exists$ a prime of the form $6k-1$ dividing it. ∎

$\boxed{\text{Prob. 1 of PSet 3}}$ $a_n :=$ # ways to pair up $2n$ players.

Ex:



$a_1 = 1$ $\quad a_2 = 3 \cdot a_1 = 3 \cdot 1 = 3$ $\quad a_3 = 5 \cdot a_2$

$$\boxed{a_n = (2n-1) \cdot a_{n-1}}$$

$\boxed{\text{Prob. 4.(b)}}$ Show $5 \mid 11^n - 6$, $\forall n \in \mathbb{N}$

Sol^n: We want to show $11^n - 6 \equiv 0 \mod 5$.

Let's see $11 \equiv 1 \mod 5$, also $6 \equiv 1 \mod 5$,

so $11^n - 6 \equiv 1^n - 1 \equiv 0 \mod 5$ $\blacksquare$