OH Wed Oct 28

Wednesday, October 28, 2020 4:04 PM

- 2a in mock + 1a
- ∞'ly primes
- 4 in mock
- 2b
- Xiyu: 1.(d).
- Prob. 4 Prac 2
- 2 on Prac 2.

Prob. 1. (a) $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$

By induction on n, the base case is $\frac{1}{\sqrt{1}} < 2\cdot 1$ ✓.

Induction step: we have $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$, we want $\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n+1}$.

Start with $$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \overset{???}{<} 2\sqrt{n+1}$$

If we prove $\boxed{2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}}$ we are done:

$$\Updownarrow$$

$$2\sqrt{n(n+1)} + 1 < \overbrace{2(n+1)}^{2n+2} \iff n(n+1) < \left[(2n+2-1)\cdot\frac{1}{2}\right]^2$$

$$\iff n(n+1) < \frac{(2n+1)^2}{4} \iff 4n^2+4n < 4n^2+4n+1 \text{ true ✓.}$$

Prob. 1.(b) $n \geq 6$ $\quad 5(n+1) \leq n^2$.

Soln: By induction on n, base case $n=6$: $5\cdot(6+1) \leq 6^2$ ✓ true

$5\cdot(6+1) = 5\cdot 7 = 35$, $\quad 6^2 = 36$

Induction step:

we have $5(n+1) \leq n^2$, we want $\underbrace{5(n+2)}_{\text{LHS}} \leq (n+1)^2$.

?

$$\overset{\text{LHS}}{\overset{\parallel}{5(n+2)}} = 5(n+1)+5 \leq n^2+5 \overset{???}{<} (n+1)^2.$$

$$5(n+2) = 5(n+1)+5 \leqslant n^2+5 < (n+1)^2.$$

Need to check $n^2+5 \leqslant \underbrace{(n+1)^2}_{= n^2+2n+1} \iff 5 \leqslant 2n+1$ true if $n \geqslant 6$. ∎

Prob. 2(a). : $x_{n+1} = x_n + (n-1)$, $x_1 = 19$. Find x_{2020}.

Sol^n: Note that $x_{n+1} = x_n + (n-1) = x_{n-1} + (n-2) + (n-1) = \cdots$

$$\cdots = \left(\sum_{k=2}^{n-1} k\right) + x_2 = \left(\sum_{k=1}^{n-1} k\right) + x_1 = \left(\sum_{k=1}^{n-1} k\right) + 19 =$$

$$= \binom{n-1}{2} + 19 = \frac{n\cdot(n-1)}{2} + 19.$$

So $\boxed{x_{2020} = \frac{2020 \cdot 2019}{2} + 19} = 2039209.$ ∎

2.(b). (Friday see Lect. on Oct 16) $x_0 = 2$, $x_1 = 3$, $\boxed{x_n = 7x_{n-1} - 10x_{n-2}}$.

r^2 r^1 r^0

Sol^n: By char. poly. method $p(r) = r^2 - 7r + 10$,

which factorizes as $p(r) = (r-5)(r-2)$.

Hence $x_n = C\cdot 5^n + D\cdot 2^n$.

Need to find C & D by using $x_0 = 2$, $x_1 = 3$

$$\left.\begin{aligned} 2 &= C + D \\ 3 &= 5C + 2D \end{aligned}\right\} \rightsquigarrow \text{solve for } C \,\&\, D.$$

Prob. 4(a): Show $\nexists\, x, y \in \mathbb{Z}$ s.t. $\boxed{4x^3 - 7y^3 = 2003.}$

Sol^n: let's try mod 3: $x^3 - y^3 \equiv 2 \mod 3.$ $\rightsquigarrow$ 3 doesn't work

by Fermat

$\overset{2}{x} - \overset{0}{y} \equiv 2 \mod 3$ $\rightsquigarrow$ ooops! $\exists$ sol^n!

Let's try modulo 7:

$4\underbrace{x^3}_{0,\,1,\,-1} \equiv 1 \mod 7.$

$97 \left(\begin{array}{l} 2100 \equiv 0 \\ 2003 \end{array} \right.$

Can this be solved?

Let's understand cubes mod 7:

$$\left.\begin{aligned} 0^3 &\equiv 0 \\ 1^3 &\equiv 1 \\ 2^3 &\equiv 1 \\ 3^3 &\equiv -1 \\ 4^3 &\equiv 1 \\ 5^3 &\equiv -1 \\ 6^3 &\equiv -1 \end{aligned}\right\} \text{possible cubes are } 0, \pm 1. \implies \text{not possible that } 4 \cdot (\text{cube}) \equiv 1 \mod 7.$$

$\blacksquare$

Prob. 4(b): 62^{48} last 2 digits?

$24^4 = 3^4 \cdot 2^{12} = 81 \cdot 4096$

$24 = 3 \cdot 8$ $\qquad = 81 \cdot 96$

Sol^n: What is $62^{48} \equiv x \mod 100$, with $0 \leq x \leq 99$?

$48 = 40 + 8$

$62 = 2 \cdot 31 \rightsquigarrow \tilde{2}^{48} \equiv 2^8 (\underbrace{2^{10}}_{1024})^4 \equiv 2^8 \cdot 24^4 \equiv 56 \cdot 24^4 \equiv 56 \cdot 76 \equiv 56.$

$62 = 2 \cdot 31 \rightsquigarrow 2 = 2(\underbrace{2}_{1024}) = 2 \cdot 24 \equiv 56 \cdot 24 \equiv 56 \cdot 16 \equiv 56.$

$\rightsquigarrow 31^{48} \equiv$ similarly break the 48.
(try see patterns for 31)

<u>Prob. on ∞'ly many primes:</u> of the form $4K-1$.

<u>Sol^n</u>: By contradiction, assume $\exists$ finitely many primes of the form $4k-1$.
Call them $\{p_1, \dots, p_N\}$. Now consider

$$\boxed{S := 4\,(p_1 \cdots p_N) - 1}$$

(i) Show that S must be divisible by a prime of the form $4K-1$.
(ii) Show that any prime of the form $4K-1$ does not divide S.

as usual

if both true then contradiction

For (i) S is divisible by primes $q_1, q_2, \dots, q_\ell$.

First, q_i must be $4k+1$ or $4k-1$, for all $1 \leq i \leq \ell$.

If q_i were all $4k+1$, then $q_1 \cdots q_\ell = S$ would be $4k+1$.

$\underset{1}{\underset{\equiv}{q_1}} \cdots \underset{1}{\underset{\equiv}{q_\ell}} = \underset{1}{\underset{\equiv}{S}}$ mod 4.

Since $S \equiv -1 \bmod 4$, not 1,
not all of the q_i can be $4k+1$, So $\exists$ at least one q_i of the form $4k-1$.

∎

Prob. 1.(d) $6 \mid k^3 - k, \ \forall k \in \mathbb{N}$.

Sol^n: We want $k^3 \equiv k \mod 6$.

Since modulo 6 we only have numbers $0, 1, 2, 3, 4, 5$.

We check: modulo 6

$$\left.\begin{array}{l} 0^3 \equiv 0 \\ 1^3 \equiv 1 \\ 2^3 \equiv 8 \equiv 2 \\ 3^3 \equiv 27 \equiv 3 \\ 4^3 \equiv 64 \equiv 4 \\ 5^3 \equiv (-1)^3 \equiv -1 \equiv 5 \end{array}\right\} \Rightarrow k^3 \equiv k \mod 6.$$

$\blacksquare$

Prob. 4.(ii) $5 \mid 11^n - 6$

Sol^n: we want is $11^n - 6 \equiv 0 \mod 5$.

Now $11 \equiv 1 \mod 5$

$6 \equiv 1 \mod 5$

So $11^n \equiv 1 \mod 5$ and

$11^n - 6 \equiv 1 - 1 \equiv 0 \mod 5$. $\blacksquare$

Prob. 2.(a): $3 \nmid 4^{100}$.

Sol^n: we want $4^{100} \equiv x \mod 3$, $0 \leq x \leq 2$

and check $x \not\equiv 0 \mod 3$.

$$4^{100} \equiv 4^{2 \cdot 50} \equiv (4^2)^{50} \equiv 1^{50} \equiv 1 \mod 3. \quad \blacksquare$$

$1 \not\equiv 0$

$4^2 \equiv 16 \equiv 1 \mod 3$

(or

$4 \equiv 1 \mod 3$

so $4^{100} \equiv 1 \mod 3$)

$\boxed{\text{Prob. 2(b)}}$ $9 \mid 10^{n+1} + 9 \cdot n^2 + 4 \cdot 10^n - 5, \forall n$

Sol^n: we want $10^{n+1} + 9 \cdot n^2 + 4 \cdot 10^n - 5 \equiv 0 \mod 9$.

For that, note $\left.\begin{array}{l} 10 \equiv 1 \mod 9 \\ 9 \equiv 0 \mod 9 \end{array}\right\}$ so $10^{n+1} + 9 \cdot n^2 + 4 \cdot 10^n - 5 \equiv 1^{n+1} + 0 + 4 \cdot 1^n - 5$

$\equiv 1 + 4 - 5 \equiv 0$ modulo 9. $\blacksquare$