MAT 108: PROBLEM SET 1

ABSTRACT. This problem set corresponds to the first week of the course Sep 30-Oct 2. It was posted online on Tuesday Sep 29 and is due Monday Oct 9 at the beginning of the class at 9:00am.

Purpose: The goal of this assignment is to practice the REP principle:

Read the statement, **E**xperiment with examples and **P**rove or disprove the statement.

In order to do that, you will be thinking about mathematical statements and trying to decide whether they are true or false. Mathematical statements are called *Lemmas*, *Propositions* and *Theorems* if they are correct. The choice of which of the three names is given is rather arbitrary, but the general rule is that Lemmas are easier to prove than Propositions, which at the same time are easier to prove than Theorems.

The three steps of the REP principle ask you to read the statement as many times as needed so that you fully understand it. The second step is that you come up with a few examples to verify whether the statement could be true or it is false. Finally, in case you believe the statement is true you might try and prove it. If you believe the statement is false, you must try to disprove it with a (counter)example.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded. I encourage you to think and work on Problems 8 and 9, they will not be graded but you can also learn a lot from them. Either of the first 8 Problems might appear in the exams.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. It is freely available on the authors' website and links can be found in the course website. Please contact me *immediately* if you have not been able to get a copy.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Read the following statements and show that they are *false* by giving a counter-example.

(a) Let $n, m \in \mathbb{N}$ be two natural numbers, such that n is even and m is odd. Then $n + m$ is even.

(b) Let $n \in \mathbb{N}$ be an even natural number, then there exists an even natural number $m \in \mathbb{N}$ such that their sum $n + m$ is odd.

(c) Let $a, b, c \in \mathbb{N}$ be three non-zero natural numbers such that $a^2 + b^2 = c^2$. Then the three numbers must be $a = 3$, $b = 4$ and $c = 5$.

(d) Let $\alpha, \beta, \gamma \in [0, 2\pi)$ be the three angles. There exists a unique planar triangle whose interior angles are α, β and γ.

Problem 2. Read Sections 1.1 and 1.2 in the textbook, and carefully follow their proofs of Proposition 1.6 and Proposition 1.9. Prove, using the five Axioms in Section 1.1 (and Prop. 1.6 if need) the following two propositions:

Proposition (Proposition 1.7). If m is an integer, then $0 + m = m$ and $1 \cdot m = m$.

Proposition (Proposition 1.8). If m is an integer, then $(-m) + m = 0$.

Problem 3. (10+10 pts) Let us take Axioms 1.1 through 1.5 in Section 1.1 as true, and assume Propositions 1.6 through 1.9 have been proven. Prove the following two propositions:

Proposition (Proposition 1.10). Let $m, x_1, x_2 \in \mathbb{Z}$. If m, x_1, x_2 satisfy the equations $m + x_1 = 0$ and $m + x_2 = 0$, then $x_1 = x_2$.

Proposition (Proposition 1.12). Let $x \in \mathbb{Z}$. If x has the property that for each integer m, $m + x = m$, then $x = 0$.

Problem 4. (20 pts) Discuss the difference between the following two statements and prove that at least **one** of them is false.

Statement (1). There exists a natural number $a \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have that $n + a = 7$.

Statement (2). For all $n \in \mathbb{N}$, there exists a natural number $a \in \mathbb{N}$ such that we have that $n + a = 7$.

Here 10 points are given for correctly pointing out the difference in the mathematical content between Statements 1 and 2, and 10 points are given for correctly proving that one of the statements is wrong.

Problem 5. (20 pts) Let us assume the following two *axioms*, as discussed in class:

A1. The area of a planar rectangle of sides $a, b \in \mathbb{R}$ is the product $a \cdot b$.
A2. The area of two planar figures which intersect at most along edges is the sum of the areas of each of the planar figures.

From the two Axioms A1 and A2 above, deduce that the area of a triangle with height $h \in \mathbb{R}$ and base $a \in \mathbb{R}$ equals the quantity $(a \cdot h)/2$.

Hint: try to cut the triangle in pieces and reassemble them to get a rectangle, then apply Axiom 2. Make sure to explain where are you using Axiom 2 in this argument.

Problem 6. (20 pts) Following the Axioms in Problem 5, show that a trapezoid with height $h \in \mathbb{R}$ and two horizontal basis of length $b_1, b_2 \in \mathbb{R}$ has area $h \cdot (b_1 + b_2)/2$.
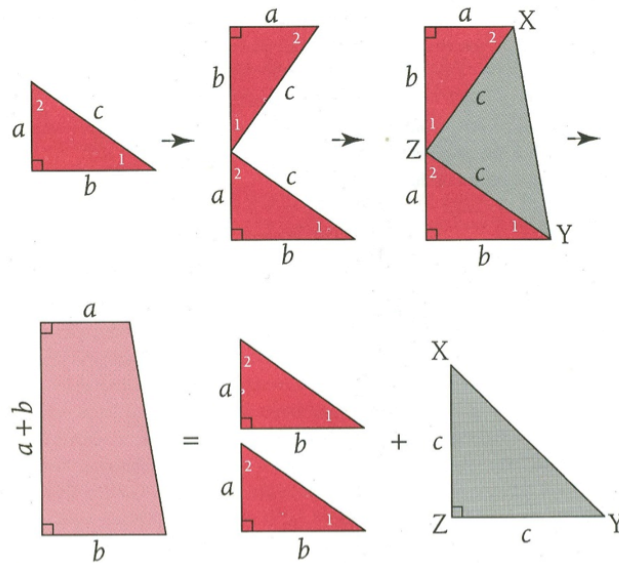

FIGURE 1. Setup for the proof of the Phytagorean Theorem as describe in Problem 7.

Problem 7. (20 pts) Let us prove the Phytagorean Theorem. Let T be a triangle with sides of length $a, b, c \in \mathbb{R}$ such that the interior angle between the a-side and the b-side is 90 degrees (a right angle). An example of such a triangle T is depicted in the upper-left corner of Figure 1. Your task is to show that

$$a^2 + b^2 = c^2.$$

This is called the Phytagorean Theorem. Here are the two steps that you might want to follow:

Step 1. (5pts) Construct the trapezoid of height $(a+b) \in \mathbb{R}$ and basis of length $a, b \in \mathbb{R}$, as in the lower left corner of Figure 1. Use the formula from Problem 6 to show that its area is $(a + b)(a + b)/2$.

Step 2. (10pts) Use the decomposition of this trapezoid as a union of triangles to show that the area of this trapezoid is also

$$2 \cdot (ab)/2 + c^2/2.$$

Step 3. (5pts) Using the Axioms A1 and A2, prove that the two areas that you have computed above must be equal and deduce that

$$a^2 + b^2 = c^2.$$

Fun fact: This proof of the Phytagorean Theorem was published in 1876 in the *New-England Journal of Education* by James A. Garfield, the 20th President of the United States. It is different from Phytagoras' proof, dating back to 500BC.

Problem 8. Consider the following sums:

$$S_0 = 0,$$
$$S_1 = \frac{1}{2},$$
$$S_2 = \frac{1}{2} + \frac{1}{4},$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$$
$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16},$$
$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32},$$
$$S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64},$$

where each sum S_n is obtained from the previous one by adding the fraction $1/2^n$.
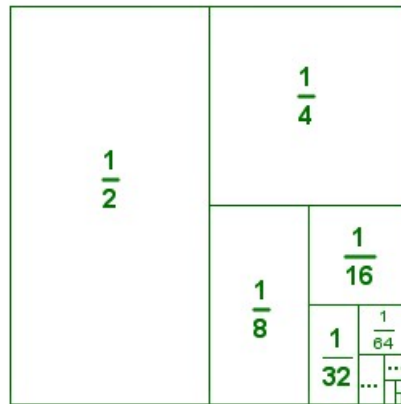

FIGURE 2. This is the Hint for Problem 8.

a. Evaluate the sums $S_1, S_2, \ldots, S_6, S_7, S_8$ and S_9.
   What do you observe when you compute these sums ?

b. Make a prediction of the approximate value of S_{100}. How about S_{10000} ?

Hint: Look at Figure 2.

Problem 9. (Optional) Explore the world from a scientific lens – on Campus, at home, wherever you are – and try to describe mathematically something you see and like. Whatever it is, keep it simple. Examples of things I like are rainbows, doors, how a baskteball spins, European stock prices or why you see a little cusp in the coffee mug when the light reflects on the surface of the coffee as in Figure 3. The following three steps might be helpful:

(1) Formulate a precise statement about the object of study,

(2) Phrase the statement in mathematical terms,

(3) Prove of disprove your statement.

FIGURE 3. This cusp appears due to the way rays bounce of the surface.