MAT 108: PROBLEM SET 4

DUE TO FRIDAY NOV 13 2020

Abstract. This problem set corresponds to the sixth week of the course, covering material on the real numbers (Chapter 8). It was posted online on Wednesday Nov 4 and is due Friday Nov 13.

Purpose: The goal of this assignment is to practice problems on the set of **real numbers**, corresponding to the sixth week of the course. This particular problem set covers the material in Chapter 8, with a view towards Chapter 10.

Task: Solve Problems 1 through 6 below. The first problem will not be graded but I trust that you will work on it. Problems 2 to 6 will be graded.

Textbook: We use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Problem 1. (Proposition 8.53) Prove that every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Problem 2. (20 points, 5 each) Find the least upper bound $\sup(A)$, and the greatest lower bound $\inf(A)$ of the following subsets of the real numbers $\mathbb{R}$:

(a) $A = (-3.2, 7) \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x < 7\} \subseteq \mathbb{R}$.

(b) $B = (-3.2, 7] \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x \leq 7\} \subseteq \mathbb{R}$.

(c) $C = (0, \infty) \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : 0 < x\} \subseteq \mathbb{R}$.

(d) $D = (-\infty, 4] \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : x \leq 4\} \subseteq \mathbb{R}$.

Problem 3. (10+10 points) Consider the set of real numbers

$$N = \left\{ 3 - \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Find $\inf(A)$ and $\sup(A)$.

Problem 4. Consider the two following subsets of the real numbers

$$S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}, \quad T = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

Show that $\sup(S) = 1$, $\sup(T) = 2$ and $\inf(T) = 3/2$. Find $\inf(S)$.

Problem 5. (10+5+5 points) Find an upper bound for each of the following three sets:

$$X = \left\{ \left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N} \right\}, \quad Y = \left\{ \left(1 + \frac{1}{n^2}\right)^n : n \in \mathbb{N} \right\}, \quad Z = \left\{ \left(1 + \frac{1}{n}\right)^{n^2} : n \in \mathbb{N} \right\}.$$

Hint: Consider the following expansion

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right).$$

Problem 6. (10+10 points) Consider the subset $C_0 = [0,1] \subseteq \mathbb{R}$. Recursively, define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right),$$

for $n \geq 1$, where, if we let $A = [a,b]$, then the notation $A/3$ describes the interval $[a/3, b/3]$ and the notation $A + 2/3$ describe the interval $[a + 2/3, b + 2/3]$.

(a) Describe and draw the sets C_1, C_2, C_3 and C_4 as a union of explicit intervals.

(b) Show that the intersection $\cap_{n=1}^{\infty} C_n$ is non-empty.