MAT 108: PROBLEM SET 5

DUE TO FRIDAY NOV 20 2020

Abstract. This problem set corresponds week of the course covering material on limits of sequences of real numbers (Chapter 10). It was posted online on Wednesday Nov 11 and is due Friday Nov 20.

Purpose: The goal of this assignment is to practice problems on **limits of sequences** of real numbers, corresponding to the weeks of the course from Friday Nov 6 to Friday Nov 13. This particular problem set covers the material in Chapter 10 of our textbook.

Task: Solve Problems 1 through 6 below. The first problem will not be graded but I trust that you will work on it. Problems 2 to 6 will be graded.

Textbook: We use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Problem 1. (Project 10.20 in textbook) Let (x_n) be a sequence of real numbers which is decreasing and bounded below. Show that x_n converges.

Problem 2. (20 points, 5 each) Consider the following four sequences of real numbers:

$$x_n = \frac{2n+1}{3n-4}, \quad y_n = \frac{1}{n!}, \quad z_n = \frac{n!}{n^n}, \quad w_n = \frac{3n^2-1}{n^2+n}.$$

In this exercise, you must use the ε-definition of the limit (Definition in Section 10.4) to show the following statements.

(a) Show that $\lim_{n\to\infty} x_n = 2/3$,

(b) Show that $\lim_{n\to\infty} y_n = 0$,

(c) Show that $\lim_{n\to\infty} z_n = 0$,

(d) Show that $\lim_{n\to\infty} w_n = 3$.

In each of these four cases above, you must write a complete detailed and self-contained proof that the limit is the one stated. Each can be done directly from the definition.

Be **clear** in the use of ε, the quantifiers and the indices when you write the four proofs above. In particular, write clearly what you are given and what you must prove when writing down the definition of each of the limits.

Problem 3. (10+10 points) Consider the following two sequences of real numbers

$$x_n = \frac{4n-3}{2^n}, \quad y_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}.$$

In this exercise we will show that they are convergent.

(a) Show that (x_n) is eventually decreasing and bounded below. By eventually decreasing it is meant that

$$x_{n+1} \leq x_n, \quad \text{for large enough } n \in \mathbb{N}.$$

(b) Show that (y_n) is increasing and bounded above.

Observation: By the Monotone Convergence Limit, you have proven that the limit of (y_n) actually exists. It is a real challenge to show that it is actually $\pi^2/6$.

Problem 4. (10+10 points) Consider the following sequence, defined recursively:

$$x_{n+1} = \frac{x_n}{2} + 1, \quad \forall n \in \mathbb{N}, \quad x_1 = 1.$$

(a) Show that (x_n) is increasing and bounded above.

(b) Prove that (x_n) converges and find its limit.

Problem 5. (5+10+5 points) Prove the following three statements:

(a) Any convergent sequence (x_n) is bounded.

(b) Let (x_n) and (y_n) be two convergent sequences, and suppose that their limits are $x_n \longrightarrow L$ and $y_n \longrightarrow M$. Show that the sequence $(x_n + y_n)$, obtained by summing them termwise, is a convergent sequence, and in fact

$$x_n + y_n \longrightarrow (L + M).$$

(c) There exist bounded sequences which are not convergent.

Problem 6. (5+5+5+5 points) Prove or disprove each of the statements.[1]

(a) The sequence $x_n = (-1)^n$ is convergent.

(b) The sequence $x_n = \frac{(-1)^n}{n}$ is convergent.

(c) Let (x_n) be a sequence such that the sequence $|x_n|$ of absolute values converges. Then (x_n) converges.

(d) Let (x_n) and (y_n) be two unbounded sequences. Then the product sequence $(x_n \cdot y_n)$ is unbounded.

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[1]If your claim is that a statement is false, then you must give a counter-example or an argument showing that the statement is indeed mathematically incorrect.