MAT 108 PRACTICE PROBLEMS I

SOLUTION SKETCHES

Note: *The following are sketches of solutions for the Practice Problems I for MAT-108, as informally written by our Reader. The writing of the solutions is not up to the standards of an explanation, and what is written in this document is only meant to provide a sketch of how a possible proof works. Please, we refer you to the Solutions of the Problem Sets 1,2 and 3, as well as the announcements on Canvas, for the correct standards of a written solutions, to which you must adhere. Still, we find that there might be some value in having these sketches so that you can practice towards your Midterm Exam this upcoming Friday October 30.*

(1) Prove the following formula for sums:

(a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Solution. Base Case: $n = 1$

$$\sum_{k=1}^{1} k = \frac{1(1+1)}{2} \qquad (1)$$
$$1 = 1$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

$$\begin{aligned} \sum_{k=1}^{j+1} k &= \sum_{k=1}^{j} k + (j+1) \\ &= \frac{j(j+1)}{2} + (j+1) \\ &= \frac{j(j+1) + 2(j+1)}{2} \\ &= \frac{(j+1)(j+2)}{2} \end{aligned} \qquad (2)$$

(b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Solution. Base Case: $n = 1$

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$$\sum_{k=1}^{1} k = \frac{1(1+1)(2(1)+1)}{6} \qquad (3)$$
$$1 = 1$$

Inductive Step: Assume true for $n=j$. Prove for $n=j+1$.

$$\begin{aligned}
\sum_{k=1}^{j+1} k^2 &= \sum_{k=1}^{j} k^2 + (j+1) \\
&= \frac{j(j+1)(2j+1)}{6} + (j+1)^2 \\
&= \frac{(j+1)(j(2j+1)+6(j+1))}{6} \\
&= \frac{(j+1)(2j^2+7j+6)}{6} \\
&= \frac{(j+1)(j+2)(2j+3)}{6}
\end{aligned} \qquad (4)$$

(c) $\sum_{k=1}^{n} k^3 = \frac{k^2(k+1)^2}{4}$

Solution.

Base Case: $n=1$

$$\sum_{k=1}^{1} k^3 = \frac{1^2(1+1)^2}{4} \qquad (5)$$
$$1 = 1$$

Inductive Step: Assume true for $n=j$. Prove for $n=j+1$.

$$\begin{aligned}
\sum_{k=1}^{j+1} k^3 &= \sum_{k=1}^{j} k^3 + (j+1)^3 \\
&= \frac{j^2(j+1)^2}{4} + (j+1)^3 \\
&= \frac{(j+1)^2(j^2+4(j+1))}{4} \\
&= \frac{(j+1)^2(j+2)^2}{4}
\end{aligned} \qquad (6)$$

(d) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Solution.

Base Case: $n=1$

(7)
$$\sum_{k=1}^{1} k^4 = \frac{1(1+1)(2(1)+1)(3(1)^2+3(1)-1)}{30}$$
$$1 = 1$$

Inductive Step: Assume true for $n = j$. Prove for $n = j+1$.

(8)
$$\begin{aligned}\sum_{k=1}^{j+1} k^4 &= \sum_{k=1}^{j} k^4 + (j+1)^4 \\ &= \frac{j(j+1)(2j+1)(3j^2+3j-1)}{30} + (j+1)^4 \\ &= \frac{(j+1)(j(2j+1)(3j^2+3j-1)+30(j+1)^3)}{30} \\ &= \frac{(j+1)(j+2)(2j+3)(3j^2+9j+5)}{30} \\ &= \frac{(j+1)(j+2)(2j+3)(3(j+1)^2+3(j+1)-1)}{30}\end{aligned}$$

(2) Prove the following additional following formulas for sums.

(a) $\sum_{k=0}^{n}(2k+1) = (n+1)^2$

Solution. Base Case: $n = 1$

(9)
$$\sum_{k=0}^{1}(2k+1) = (1+1)^2$$
$$1 + 3 = 4$$

Inductive Step: Assume true for $n = j$. Prove for $n = j+1$.

(10)
$$\begin{aligned}\sum_{k=0}^{j+1}(2k+1) &= \sum_{k=0}^{j}(2k+1) + 2(j+1) + 1 \\ &= (j+1)^2 + 2(j+1) + 1 \\ &= (j^2+2j+1) + (2j+2) + 1 \\ &= j^2 + 4j + 4 \\ &= (j+2)^2\end{aligned}$$

(b) $\sum_{k=1}^{n} 2k = n(n+1)$

Solution.

Base Case: $n = 1$

$$\begin{aligned}\sum_{k=1}^{1} 2k &= 1(1+1) \\ 2 &= 2\end{aligned} \tag{11}$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

$$\begin{aligned}\sum_{k=1}^{j+1} 2k &= \sum_{k=1}^{j} 2k + 2(j+1) \\ &= j(j+1) + 2(j+1) \\ &= (j+1)(j+2)\end{aligned} \tag{12}$$

(c) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

Solution.

Base Case: $n = 1$

$$\begin{aligned}\sum_{k=1}^{1} k(k+1) &= \frac{1(1+1)(1+2)}{3} \\ 2 &= 2\end{aligned} \tag{13}$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

$$\begin{aligned}\sum_{k=1}^{j+1} k(k+1) &= \sum_{k=1}^{j} k(k+1) + (j+1)(j+2) \\ &= \frac{j(j+1)(j+2)}{3} + (j+1)(j+2) \\ &= \frac{(j+1)(j+2)(j+3)}{3}\end{aligned} \tag{14}$$

(d) $\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$

Solution.

Base Case: $n = 1$

$$\begin{aligned}\sum_{k=0}^{1} 3^k &= \frac{3(1+1)-1}{2} \\ 1+3 &= 4\end{aligned} \tag{15}$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

$$\begin{aligned} \sum_{k=0}^{j+1} 3^k &= \sum_{k=0}^{j} 3^k + 3^{j+1} \\ &= \frac{3^{(j+1)} - 1}{2} + 3^{(j+1)} \\ &= \frac{3 \cdot 3^{(k+1)} - 1}{2} \\ &= \frac{3^{(k+2)} - 1}{2} \end{aligned} \tag{16}$$

(e) $\sum_{k=1}^{n} k!k = (n + 1)! - 1$

Solution.

Base Case: $n = 1$

$$\begin{aligned} \sum_{k=1}^{1} k!k &= (1 + 1)! - 1 \\ 1 &= 1 \end{aligned} \tag{17}$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

$$\begin{aligned} \sum_{k=1}^{j+1} k!k &= \sum_{k=1}^{j} k!k + (j + 1)!j \\ &= (j + 1)! - 1 + (j + 1)!(j + 1) \\ &= (j + 1)!(1 + j + 1) - 1 \\ &= (j + 2)! - 1 \end{aligned} \tag{18}$$

(3) Prove the following inequalities. Be aware of the base case in each case.

(a) For all $n \in \mathbb{N}, n < 2^n$

Solution.

Base Case: $n = 1$

$$\begin{aligned} 1 &< 2^1 \\ 1 &< 2 \end{aligned} \tag{19}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$\begin{aligned} k &< 2^k \\ 2k &< 2^{k+1} \\ k + 1 \leq 2k &< 2^{k+1} \end{aligned} \tag{20}$$

(b) For all $n \in \mathbb{N}, n^2 + 6n + 7 < 20n^2$

Solution.

Base Case: $n = 1$

$$\begin{aligned} 1^2 + 6(1) + 7 &< 20(1)^2 \\ 14 &< 20 \end{aligned} \tag{21}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$\begin{aligned} (k+1)^2 + 6(k+1) + 7 &< 20(k+1)^2 \\ k^2 + 2k + 1 + 6k + 1 + 7 &< 20k^2 + 40k + 20 \\ k^2 + 8k + 9 &< 20k^2 + 40k + 20 \\ k^2 + 8k + 9 < 20k^2 &< 20k^2 + 40k + 20 \end{aligned} \tag{22}$$

(c) For $n \geq 4, n^2 \leq 2^n$

Solution.

Base Case: $n = 4$

$$\begin{aligned} 4^2 &\leq 2^4 \\ 16 &\leq 16 \end{aligned} \tag{23}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$\begin{aligned} (k+1)^2 &< 2^{k+1} \\ 2k^2 &< 2 \cdot 2^k \\ (k+1)^2 < 2k^2 &< 2^{k+1} \end{aligned} \tag{24}$$

(d) For $n \geq 4, 2^n < n!$

Solution.

Base Case: $n = 4$

$$\begin{aligned} 2^4 &< 4! \\ 16 &< 24 \end{aligned} \tag{25}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

(26)
$$\begin{aligned} 2^k &< k! \\ (k+1)\cdot 2^k &< (k+1)! \\ (2+(k-1))\cdot 2^k &< (k+1)! \\ 2^{k+1} + 2^k\cdot(k+1) &< (k+1)! \\ 2^{k+1} < 2^{k+1} + 2^k\cdot(k+1) &< (k+1)! \end{aligned}$$

(e) For $n \geq 6, 6(n+1) < 2^n$

Solution.

Base Case: $n = 6$

(27)
$$\begin{aligned} 6(6+1) &< 2^6 \\ 42 &< 64 \end{aligned}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k+1$.

(28)
$$\begin{aligned} 6k+6 &< 2^k \\ 12k+12 &< 2^{k+1} \\ 6k+12 &< 2^{k+1} \\ 6(k+2) < 12k+12 &< 2^{k+1} \end{aligned}$$

(f) For $n \geq 8, 3n^2+3n+1 < 2^n$

Solution.

Base Case: $n = 8$

(29)
$$\begin{aligned} 3(8)^2+3(8)+1 &< 2^8 \\ 217 &< 256 \end{aligned}$$

Inductive Step: Assume true for $n = k$. Prove $n = k+1$.

(30)
$$\begin{aligned} 2(3k^2+3k+1) &< 2^{k+1} \\ 3(k-1)^2+3(k+1)+1 &< 2^{k+1} \\ 3(k-1)^2+3(k+1)+1 < 2(3k^2+3k+1) &< 2^{k+1} \end{aligned}$$

(g) For $n \geq 12, 5^n < n!$

Solution.

Base Case: $n = 12$

$$\begin{aligned} 5^{12} &< 12! \\ 244140625 &< 479001600 \end{aligned} \tag{31}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$\begin{aligned} 5^k &< k! \\ (k+1)5^k &< (k+1)! \\ (5 + (k-4))5^k &< (k+1)! \\ 5^{k+1} + (k-4)5^k &< (k+1)! \\ 5^k < 5^{k+1} + (k-4)5^k &< (k+1)! \end{aligned} \tag{32}$$

(4) Show that the following divisibility statements are true.

(a) For all $n \in \mathbb{N}, 4|(5^n - 1)$

Solution.

Base Case: $n = 1$

$$5^1 - 1 \equiv 0 \pmod 4 \tag{33}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$. Let $4x = 5^k - 1$, so $5^k = 4x + 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5 \cdot 5^k - 1 \\ &= 5(4x + 1) - 1 \\ &= 20x + 5 - 1 \\ &= 4(5x + 1) \end{aligned} \tag{34}$$

(b) For all $n \in \mathbb{N}, 5|(11^n - 6)$

Solution.

Base Case: $n = 1$

$$11^1 - 6 \equiv 0 \pmod 5 \tag{35}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$. Let $5x = 11^k - 6$, so $11^k = 5x + 6$.

$$\begin{aligned} 11^{k+1} - 6 &= 11 \cdot 11^k - 6 \\ &= 11(5x + 6) - 6 \\ &= 55x + 60 \\ &= 5(11x + 12) \end{aligned} \tag{36}$$

(c) For all $n \in \mathbb{N}, 6|(n^3 - n)$

Solution.

Base Case: $n = 1$

(37) $$1^3 - 1 \equiv 0 \pmod 6$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$. Let $6x = k^3 - k$.

(38) $$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= k^3 + 3k^2 + (3k - k) \\ &= 3x + 3k^2 + 3k \\ &= 3(x + k^2 + k)\end{aligned}$$

(d) For all $n \in \mathbb{N}, 7|(2^{n+2} + 3^{2n+1})$

Solution.

Base Case: $n = 1$

(39) $$2^{1+2} + 3^{2n+1} = 8 + 27 \equiv 0 \pmod 7$$

Inductive Step: Assume true for $n = k$. Prove for $n = k+1$. Let $7x = 2^{k+2} + 32k + 1$, so $2^{k+2} = 7x - 3^{2k+1}$.

(40) $$\begin{aligned}2^{k+3} + 3^{2k+3} &= 2 \cdot 2^{k+2} + 3^{2k+3} \\ &= 2(7x - 3^{2k+1}) + 3^{2k+3} \\ &= 14x - 2 \cdot +3^{2k+1} + 3^{2k+3} \\ &= 14x + 3^{2k+1}(-2 + 3^2) \\ &= 14x + 7(3^{2k+1}) \\ &= 7(2x + +3^{2k+1})\end{aligned}$$

(5) Prove that there are infinitely many primes of the form $6k + 5$ with $k \in \mathbb{N}$.

Solution.

Suppose there is a finite set of primes of the form $6k+5$. Let $A := \{\text{primes of the form } 6k+5 : k \in \mathbb{N}\}$ and $P = \Pi_{i=1}^{n}(6k_i + 5)$. By the Binomial Theorem, P is in the form $6k + 5$ regardless of $|A|$. Consider the number $P + 6$, which does not change the residue. Then by **Lemma 2.1**, since $6k_i + 5 \nmid 6$, this new number is either a new prime, or composite number composed of primes oringally A.

(6) Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 1$.

Solution.

Suppose there were positive integer solutions a, b that satisfied $a^2 - b^2 = 1$. This means that either a^2 is even and b^2 is odd, or vice versa.

Case 1: a^2 is even.

If a^2 is even, then a must have form $2n$ such that $n \in \mathbb{N}$. Thus, $a^2 = 4n^2$. Since b^2 is odd, it has form $2m+1$ such that $m \in \mathbb{N}$, so $b^2 = 4m^2 + 4m + 1$. So, $a^2 - b^2 = 4n^2 - 4m^2 - 4m - 1$, which also equals 1. From this, we get $4n^2 - 4m^2 - 4m = 2$. The left hand side is divisible by 4, but the right hand side is not. This is a contradiction, so there are no positive integer solutions for this case.

Case 2: a^2 is odd.

If a^2 is odd, then a must have form $2n + 1$, such that $n \in \mathbb{N}$, so $a^2 = 4n^2 + 4n + 1$. Similarly, since b^2 is even, it has form $2m$ such that $m \in \mathbb{N}$, so $b^2 = 4m^2$. This means $a^2 - b^2 = 4n^2 + 4n + 1 - 4m^2$, which also equals 1. Simplifying this equation gives us $n(n + 1) = m^2$. We know that $n \geq m$, so at minimum, we can say $m(m + 1) = m^2$ or $m^2 + m = m^2$, which is a contradiction. Thus there are no positive integer solutions for this case either.

(7) Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

Solution.

Suppose there were positive integer solutions a, b that satisfied $a^2 - b^2 = 10$. This means that a^2, b^2 must either both be odd or even.

Case 1: a^2, b^2 are even

If both numbers are even, that means that a, b must be even as well. Thus a^2, b^2 must be divisible by 4. Since both numbers are divisible by 4, this means the difference of the two numbers is divisible by 4 as well. $10 \nmid 4$, so this is a contradiction.

Case 2: a^2, b^2 are odd

If both numbers are odd, this means a, b are both odd numbers so a has form $2n + 1$ and b has form $2m + 1$ where $m, n \in \mathbb{N}$. This means $a^2 = 4n^2 + 4n + 1$ and $b^2 = 4m^2 + 4m + 1$. Thus, $a^2 = b^2 = 4n^2 + 4n - 4m^2 + 4m$ which is divisible by 4. Since $1 \nmid 4$, this leads to a contradiction. There are no positve integer solutions for $a^2 + b^2 = 1$.

(8) Let $a_n = 2^n + 1$, prove that a_n satisfies the recursion

$$a_{n+1} = 2a_n - 1, a_1 = 3$$

Solution.

Base Case: $n = 2$

$$\begin{aligned} a_2 &= 2a_1 - 1 = 2^2 + 1 \\ 2(3) - 1 &= 5 \end{aligned} \tag{41}$$

Inductive Step: Assume the above equation is true for $n = k$. Prove for $n = k + 1$.

(42)
$$\begin{aligned} a_{k+1} &= 2a_k - 1 \\ &= 2(2^k + 1) - 1 \\ &= 2^{k+1} + 2 - 1 \\ &= 2^{k+1} + 1 \end{aligned}$$

(9) Let F_n be the nth Fibonacci number, defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$. Prove that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

Solution.

Base Case: $n = 1$

(43)
$$\begin{aligned} \sum_{k=1}^{1} F_k^2 &= F_1 F_2 \\ 1 &= 1 \end{aligned}$$

Inductive Step: Assume true for $n = j$. Prove for $n = j + 1$.

(44)
$$\begin{aligned} \sum_{k=1}^{j+1} F_k^2 &= \sum_{k=1}^{j} F_k^2 + F_{j+1}^2 \\ &= F_j F_{j+1} + F_{j+1}^2 \\ &= F_{j+1}(F_j + F_{j+1}) \\ &= F_{j+1} F_{j+2} \end{aligned}$$

(10) Let A_n be defined by the recursion $A_{n+1} = 2A_n + 1$, and $F_1 = \alpha$. Prove that

$$A_n = (\alpha + 1) \cdot 2^{n-1} - 1$$

Solution.

Base Case: $n = 2$

(45)
$$\begin{aligned} A_2 = 2A_1 + 1 &= (\alpha + 1) \cdot 2^{2-1} - 1 \\ 2\alpha + 1 &= (2\alpha + 2) - 1 \end{aligned}$$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$
\begin{aligned}
A_{k+1} &= 2A_k + 1 \\
&= 2((\alpha+1)\cdot 2^{k-1} - 1) + 1 \\
&= ((\alpha+1)\cdot 2^k - 2) + 1 \\
&= (\alpha+1)\cdot 2^k - 1
\end{aligned} \tag{46}
$$

(11) Let L_n be defined by the recusion $L_{n+1} = L_n + L_{n-1}$ and $L_0 = 2, L_1 = 1$. Prove that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solution.

The characteristic equation for the recursive sequence is: $x^2-x-1=0$, so the roots are $x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$. Thus the general form for our closed formula is $L_n = C\cdot(\frac{1+\sqrt{5}}{2})^n + D\cdot(\frac{1-\sqrt{5}}{2})^n$. Substituting our initial values into the equation, we get:

$$
\begin{aligned}
2 &= C + D \\
1 &= \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot C + \left(\frac{1-\sqrt{5}}{2}\right) \cdot D
\end{aligned} \tag{47}
$$

From this, we get $C = 1, D = 1$. Thus our closed formula for our recursive sequence is:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{48}$$

(12) Let a_n be defined by the recursion $a_{n+1} = 7a_n - 10a_{n-1}$ and $a_0 = 2, a_1 = 3$. Find a closed formula for a_n.

Solution.

The characteristic equation for this recursive sequence is: $x^2 - 7x + 10 = 0$, so the roots to this equation are $x = 2, 5$. Thus the general form for our closed formula is $A_n = C\cdot 2^n + D\cdot 5^n$. Substituting our points into our formula gives us:

$$
\begin{aligned}
2 &= C + D \\
3 &= 2C + 5D
\end{aligned} \tag{49}
$$

From this, we get $C = \frac{7}{3}, D = \frac{-1}{3}$. Thus our closed formula for our recursive sequence is:

$$A_n = \frac{7}{3}\cdot 2^n - \frac{1}{3}\cdot 5^n \tag{50}$$