COHERENT STATE

– quantum state obeying classical EoM

Application LASER

straight line trajectory

EXAMPLE: Harmonic Oscillator

$$H = \frac{1}{2}(p^2 + q^2)$$

$$= \hbar\left(N + \frac{1}{2}\right)$$

Where $N = a^\dagger a$, $[a, a^\dagger] = 1$

Number operator

L2

Vacuum $|0\rangle$ = Gaußian Wavepacket

Energy eigenstates

$$|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle$$

Obey $\langle n|m\rangle = \delta_{n,m}$

$$H|n\rangle = \hbar(n+\tfrac{1}{2})|n\rangle$$

Resolution of unity

$$\underset{\mathcal{H}^{*}}{\underbrace{\sum_n |n\rangle}}\underset{\mathcal{H}}{\underbrace{\langle n|}} = \mathrm{Id}$$

Let $z \in \mathbb{C}$ and consider

$$|z\rangle := e^{za^+}|0\rangle$$

which is an a-eigenstate b/c

$$a|z\rangle = [a, e^{za^+}]|0\rangle = z e^{za^+}|0\rangle = z|z\rangle$$

Unitary evolution

$$|z(t)\rangle := e^{-\frac{i}{\hbar}Ht}|z\rangle$$

$$= e^{-i(N+\frac{1}{2})t} e^{za^+}|0\rangle$$

Lemma

$$e^{-iNt} f(a^+) e^{iNt} = f(e^{-it} a^+)$$

Proof $e^{-iNt} a^+ e^{iNt} =$

$Na^+ = a^+(N+1)$

$$= a^{+} e^{-i(N+1)t} e^{-iNt} = e^{-it} a^{+} \quad [4]$$

Similarly

$$e^{-iNt} a^{+k} e^{iNt}$$

$$= \underbrace{e^{-iNt} a^{+} e^{iNt} \cdots e^{-iNt} a^{+} e^{iNt}}_{k\text{-times}}$$

$$= e^{-ikt} a^{+k} \qquad \square$$

Because $N|0\rangle = 0$,

$$e^{iNt}|0\rangle = |0\rangle$$

Thence

$$\boxed{|z(t)\rangle = e^{-\frac{it}{2}} |e^{-it} z\rangle}$$

$\Leftrightarrow$ coherent $\circlearrowright$ evolution

Classical motion $z = x + ip \Rightarrow$ $\underline{\text{LE}}$

$$x(t) = \operatorname{Re} z(t) = x\cos t + p\sin t$$

$$p(t) = \operatorname{Im} z(t) = p\cos t - x\sin t$$

Propagator

$$P_{z \overset{t}{\longmapsto} z'} = \left| \frac{\langle z' | z(t) \rangle}{\langle z' | z \rangle} \right|^2$$

The function

$$K(z, z'; t) := \langle z' | e^{-iHt/\hbar} | z \rangle$$

is called the (coherent state) propagator.

Observe

$$\frac{\partial}{\partial z} |z\rangle = \frac{\partial}{\partial z} e^{z a^\dagger} |0\rangle = a^\dagger |z\rangle$$

$$z|z\rangle = [a, e^{za^+}]|0\rangle = a|z\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}K = \langle z'|e^{-iHt/\hbar}H|z\rangle$$

$$= \langle z'|e^{-iHt/\hbar}(a^+a+\tfrac{1}{2})|z\rangle$$

$$= \left(\frac{\partial}{\partial z}z + \frac{1}{2}\right)K$$

$\Rightarrow$ The propagator K obeys the Schrödinger equation

Remark In QFT the propagator $\langle out|e^{-\frac{iHt}{\hbar}}|in\rangle$ as $t\to\infty$ is called the S-matrix

Resolution of unity

Claim

$$\mathbb{1} = \int \frac{d\bar{z}dz}{2\pi i} |z\rangle e^{-\bar{z}z} \langle z|$$

Proof: $d\bar{z}\, d\bar{z} = 2i\, dx \wedge dy$

$$= 2i\, r\, dr\, d\theta$$

$$e^{-\bar{z}z} = e^{-(x^2+y^2)} = e^{-r^2}$$

$$|z\rangle\langle z| = \sum_{n,m} \frac{\bar{z}^n z^m}{\sqrt{m!n!}} |n\rangle\langle m|$$

$$= \sum_{n,m} \frac{r^{n+m} e^{i(m-n)\theta} |n\rangle\langle m|}{\sqrt{m!n!}}$$

$$\int_0^{2\pi} d\theta\, e^{i(m-n)\theta} = 2\pi \delta_{m,n}$$

Orchestrating

$$\text{RHS} = \frac{2\pi}{\pi} \sum_n \int \frac{r^{2n+1} e^{-r^2}}{n!} |n\rangle\langle n|$$

$$= \sum_n \underbrace{\int_0^\infty \frac{u^n}{n!} e^{-u}}_{1} |n\rangle\langle n|$$

$$= \mathrm{Id} \qquad \square$$

Path integral

Claim

Quantum propagator & hence probability is obtained by integrating over all paths connecting z & z' with a suitable measure.

Idea (Feynman), split large time quantum evolution into infinitesimal pieces & concatenate.

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right) = \underbrace{e^{-\frac{i\hat{H}\Delta t}{\hbar}} \cdots e^{-\frac{i\hat{H}\Delta t}{\hbar}}}_{N \text{ times}}$$

$$\Delta t := \frac{t}{N} \xrightarrow[N\to\infty]{} 0$$

Dirac: Small time evolution controlled by classical action principle!

$$\langle z' | e^{-\frac{i\hat{H}\Delta t}{\hbar}} | z\rangle$$

$$\approx \langle z' | \left(1 - \frac{i\hat{H}\Delta t}{\hbar}\right) | z\rangle$$

$$= \langle z' | \left(1 - \frac{i(a^{+}a + \frac{1}{2})\Delta t}{\hbar}\right) | z\rangle$$

Harmonic oscillator

$$= \langle z' | z\rangle \left(1 - \frac{i(\bar{z}'z + \frac{1}{2})\Delta t}{\hbar}\right)$$

$$\approx \langle z' | z\rangle \, e^{-i(\bar{z}'z + \frac{1}{2})\Delta t}$$

exp("Liouville form")

classical Hamiltonian $p^2 + q^2$

Remark The relation

$$\langle z' | \hat{H} | z \rangle = H_{class}(z', z)$$

only holds for "free" theories

$\Rightarrow$ in general get

$$H_{class} + \mathcal{O}(\hbar)$$

"Counter-terms"

$\rightsquigarrow$ cf. Quantum Field Theory

Trick Insert resolutions of unity to concatenate Dirac result

Recall $\mathbb{1} = \int \mu_i \, |z_i\rangle e^{-\bar{z}_i z_i} \langle z_i|$

so that

$$K(z', z; t) =$$

$$\int \mu_1 \cdots \mu_{N-1} \langle z| e^{-\frac{i\hat{H}\Delta t}{\hbar}} |z_{N-1}\rangle$$

$$e^{-\bar{z}_{N-1} z_{N-1}} \langle z_{N-1}| \cdots$$

$[\mu]$

$$\langle z_i| e^{-\frac{i\hat{H}\Delta t}{\hbar}} |z_{i-1}\rangle$$

$$\times\, e^{-\bar{z}_{i-1} z_{i-1}} \cdots$$

$$e^{-\bar{z}_1 z_1} \langle z_1| e^{-\frac{i\hat{H}\Delta t}{\hbar}} |z\rangle$$

$$\approx \int [\mu] \langle z|z_{N-1}\rangle e^{-iH_{cl}(\bar{z}, z_{N-1}) - \bar{z}_{N-1} z_{N-1}}$$

$$\cdots e^{-\bar{z}_1 z_1} e^{-\frac{iH_{cl}(\bar{z}_1, z)\Delta t}{\hbar}} \langle z_1|z\rangle$$

Note that

$$\langle \bar{z}'|z\rangle = \langle 0| e^{\bar{z}' a^{+}} e^{z a} |0\rangle$$

$$= \sum_{m,n} \langle 0 | a^{+m} a^{n} | 0 \rangle \times \frac{\bar{z}'^{m} z^{n}}{m! \; n!}$$

$$= \sum_{m,n} \delta_{m,n} \, m! \, \frac{\bar{z}'^{m} z^{n}}{m! \; n!}$$

$$= \exp(\bar{z}' z)$$

Orchestrating

$$K(\bar{z}; z; t) \approx \int [\mu] \times$$

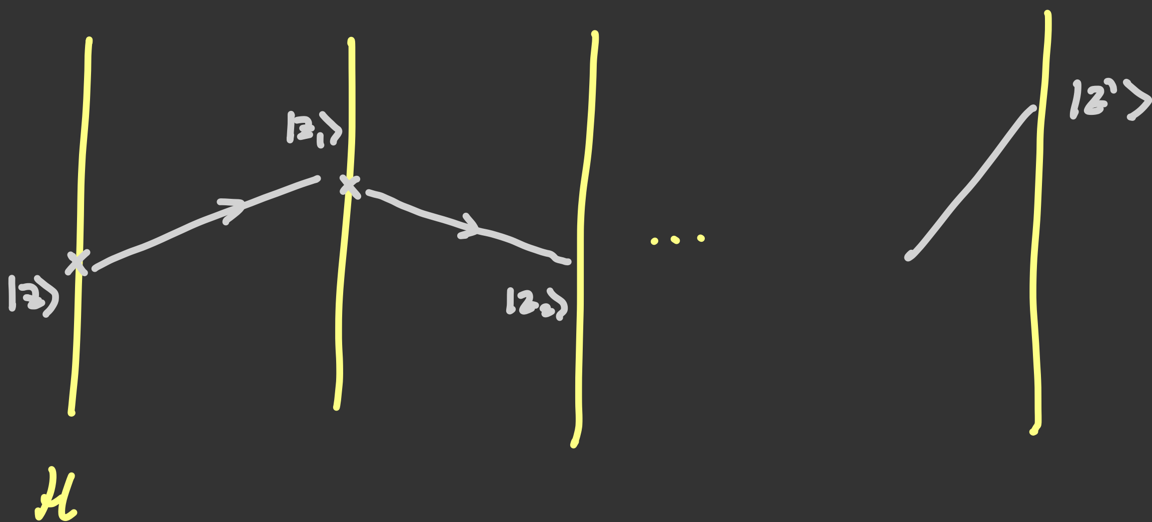
$$e^{(\bar{z} - \bar{z}_{N-1}) z - \frac{i}{\hbar} H_{cl}(\bar{z}, z_{N-1}) \Delta t}$$

$$\times\, e^{(\bar z_{N-1} - \bar z_{N-2}) z_{N-1} - \frac{i}{\hbar} H_{cl}(\bar z_{N-1}, z_{N-2}) \Delta t}$$

$$\vdots$$

$$\times\, e^{(\bar z_2 - \bar z_1) z - \frac{i}{\hbar} H_{cl}(\bar z_1, z)}\; e^{\bar z_1 z}$$

A picture



$$\underbrace{\qquad\qquad}_{\Delta t = \frac{t}{N}}$$

As $N \longrightarrow \infty$

approximate

$$z_i \approx z(\tau) \in \mathbb{C}$$

$$\tau \in [0, t].$$

Also $(\bar{z}_i - \bar{z}_{i-1}) z_i = \dfrac{\bar{z}_i - \bar{z}_{i-1}}{\Delta t} z_i \Delta t$

$$\approx \dot{\bar{z}}(\tau) z(\tau)\, d\tau$$

Thus

$$\boxed{K(z', z; t) = e^{|z|^2} \int [\mu] \exp \frac{i}{\hbar} L(z, \bar{z})}$$

integration over paths from z to z'

PATH - INTEGRAL

Where

$$L = \int_0^t \left(\frac{\hbar}{i} \dot{\bar{z}} z - H_{cl}(z(\tau), \bar{z}(\tau)) \right) d\tau$$

What does an integration over paths mean?

How to compute path integrals?

SEMI-CLASSICAL APPROXIMATION

Given Lagrangian

$L(X(t))$ where

$$X : I \longrightarrow M$$

worldline

target space

We "define" the quantization of L by a path integral

$$K_{fi} := \langle X_f | e^{\frac{-i\hat{H}t}{\hbar}} | X_i \rangle = \int_{X_i}^{X_f} [dX] \, e^{\frac{iL[X]}{\hbar}}$$

paths from X_i to X_f

Suppose that $y(t)$ extremizes L.

i.e. $\quad \frac{d}{ds} L\left(y(t)+s z(t)\right)\Big|_{s=0} = 0$

Then

$$L(y+z) = L(y) + \mathcal{O}(y^2)$$

Thus

$$K_{fi} \approx e^{\frac{iL(y)}{\hbar}} \int [dz]\, e^{i\left[L(y+z)-L(z)\right]}$$

Dirac's leading approximation

Handle perturbatively

GRAPHICAL EXPANSION

Feynman Diagrams

Consider

$$K(g) \overset{g\geq 0}{=} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2 - \frac{g}{4!}x^4}$$

Free theory

Interaction

Maple says if $g \geq 0$

$$K = \sqrt{\frac{3}{2\pi g}}\, e^{\frac{3}{4g}}\, K_{\frac{1}{4}}\left(\frac{3}{4g}\right)$$

Modified Bessel function index $\frac{1}{4}$ of 2nd kind.

$$\overset{g\to 0}{\sim} 1 - \frac{g}{8} + \frac{35}{384}g^2 - \frac{385}{3072}g^3 + \frac{25025}{98304}g^4 + \cdots$$

Maple with some coaxing ...

Graphical expansion

Study a new integral

$$Z(j,g) := \int \frac{dx}{\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2}x^2 - \frac{g}{4!}x^4}_{\text{Lagrangian}} + jx}$$

Partition function

Fields

Source

Nb: $Z(0,j) = K(g)$.

Schwinger trick:

$$\frac{d}{dj}\int \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2 + jx}\Bigg|_{j=0}$$

$$= \int \frac{dx}{\sqrt{2\pi}}\, x e^{-\frac{1}{2}x^2}$$

$$=: \bar{x}$$

Normal distribution

Two ways to see this vanishes

i) x is an odd function

ii) $$\int \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2 + jx}$$

$$= e^{\frac{1}{2}j^2} \int \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-j)^2}$$

$$= e^{\frac{1}{2}j^2}$$

and $$\frac{d}{dj} e^{\frac{1}{2}j^2}\Big|_{j=0} = 0$$

Similarly

$$\overline{x^2} = \int \frac{dx}{\sqrt{2\pi}}\, x^2 e^{-\frac{1}{2}x^2}$$

$$= \frac{d^2}{dj^2} \int \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2 + jx}\Big|_{j=0}$$

$$= \frac{d^2}{dj^2} e^{\frac{1}{2}j^2}\Big|_{j=0}$$

$$= \frac{d}{dj}\left(j e^{\frac{1}{2}j^2}\right)\Big|_{j=0}$$

$$= 1$$

Or even braver, for f analytic

$$\boxed{\overline{f(x)} = f\left(\frac{d}{dj}\right) e^{\frac{1}{2}j^2}\Big|_{j=0}}$$

Hence

$$K(g) = e^{-\frac{g}{4!}\frac{d^4}{dj^4}} e^{\frac{1}{2}j^2}\Big|_{j=0}$$

Check:

$$-\frac{1}{4!}\frac{d^4}{dj^4} e^{\frac{1}{2}j^2}\Big|_{j=0} = -\frac{1}{8}$$

$$\frac{1}{2!}\frac{1}{4!^2}\frac{d^8}{dj^8} e^{\frac{1}{2}j^2}\Big|_{j=0} = \frac{35}{384}$$

$$-\frac{1}{3!}\frac{1}{4!^2}\frac{d^{12}}{dj^{12}} e^{\frac{1}{2}j^2}\Big|_{j=0} = -\frac{385}{3072}$$

etc...

BUT WHERE ARE THE GRAPHS ??

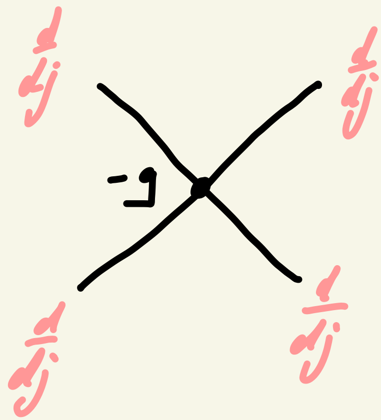
Feynman Rules

Pictorial notation

$\frac{1}{2}j^2 =$

edges

$-\frac{g}{4!}\frac{d^4}{dj^4} =$



vertices

Compute

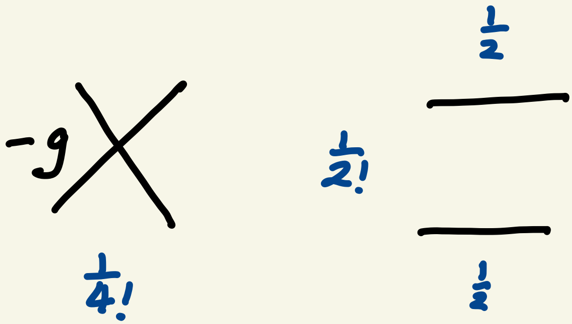
$$\exp\left(-\frac{g}{4!}\frac{d^4}{dj^4}\right)\exp\frac{1}{2}j^2\Big|_{j=0}$$

by drawing pictures.

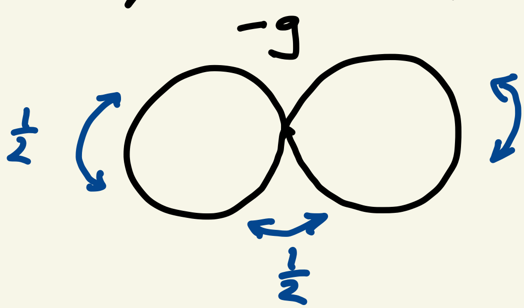
* Since we set $j=0$ must connect all vertices & edges

* Work order by order in g

g^1



Only one picture possible

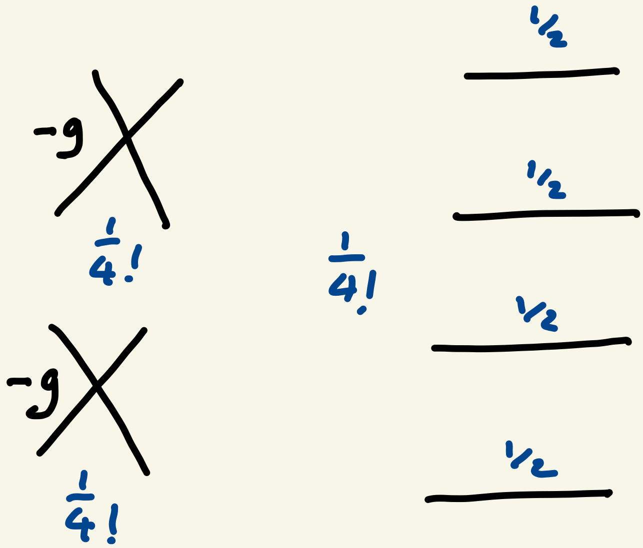


$= -\frac{g}{8}$

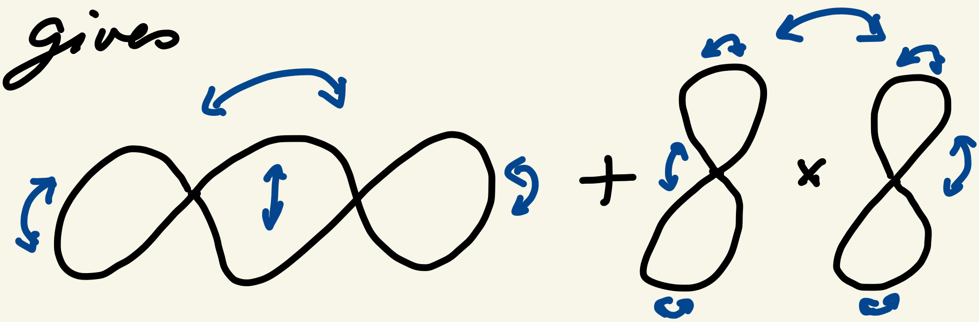
naively combinatorial factors cancel unless graph has automorphisms

$g^2 \Big|$

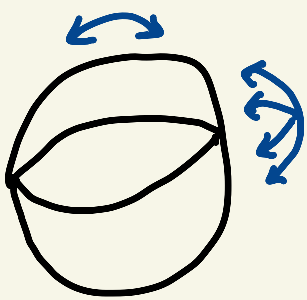
$\frac{1}{2!}$



gives



$\frac{1}{16}\, g^2$ $\qquad$ $\frac{1}{128}\, g^2$

$+$  $= \frac{35}{384}\, g^2$

$\frac{1}{48}\, g^2$

General result

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2 - \frac{g}{4!}x^4} = \sum_{\substack{\Gamma \\ 4\text{-valent}}} \frac{(-g)^{V_\Gamma}}{|\mathrm{Aut}\,\Gamma|}$$