Lecture 10

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Lecture 10 : The Heisenberg uncertainty principle

Heisenberg's Picture

need to define $\{\cdot,\cdot\}$

$$\begin{cases} \dot{f}_t = \{f_t, H\} \\ \dot{\rho}_t = 0 \end{cases} \rightsquigarrow \begin{cases} \dot{A}_t = \{A_t, H\} \\ \dot{M}_t = 0 \end{cases}$$

Schrödinger's pde

$$\begin{cases} \dot{f}_t = 0 \\ \dot{\rho}_t = -\{\rho_t, H\} \end{cases} \rightsquigarrow \begin{cases} \dot{A}_t = 0 \\ \dot{M}_t = -\{M_t, H\} \end{cases}$$

Here ρ_t gives $\mathbb{E}_{\mu_t}(\cdot) = \int_M \cdot\, \rho_t \cdot \omega^n$, μ_t is a state.

$G(M)$ algebra of observ. $\rightsquigarrow$ $G(M) := C^\infty(M)$ classically $\xrightarrow{\text{"quantizing"}}$ $G := \mathcal{H}(V)$ hermitian op. in $(V, \langle\cdot,\cdot\rangle)$ $\mathbb{C}$-v.s. w/ unitary product. $\{\cdot,\cdot\}$? what's product ?

$S(M)$ set of states $\rightsquigarrow$ ρ density fct $\longrightarrow$? (M_t)

§1. Observables being Hermitian Op. : let $(V, \langle\cdot,\cdot\rangle)$ be $\mathbb{C}$-v.s. , and consider

(a) $G(V) := \{A : V \to V : \langle Av, v\rangle = \langle v, Av\rangle\}$, A hermitian, i.e. $A^\dagger = A$.

our observables $\longrightarrow$ $\{\cdot,\cdot\}$ Poisson $\rightsquigarrow$ commutator

$\longrightarrow$ algebra (not $A\cdot B$ as composition)

$\Rightarrow$ algebra of observables

For Poisson we define $\{A,B\} := \frac{i}{\hbar}[A,B] = \frac{i}{\hbar}(AB-BA)$.

The product will be $\frac{AB+BA}{2}$. $\leftarrow$ "Jordan algebras"

(b) **States** : a lemma shows that linear functionals $\ell : \mathcal{H}(V) \longrightarrow \mathbb{C}$ (i.e. a state)

are given by $M = M(\ell)$ with $\ell(A) = tr(M_\ell \cdot A)$, w/ M non neg. def., self & $tr(M) = 1$.

$\Rightarrow$ states are $M \in End(V)$ s.t. non-neg, self-adj. & $tr(M) = 1$.

Thm : (Heisenberg) Let $A, B \in G(V)$, and μ a state. Then

$$\boxed{\sigma_\mu(A)\cdot\sigma_\mu(B) \geq \frac{\hbar}{2}\cdot\left|\mathbb{E}_\mu\left(\underbrace{\{A,B\}_\hbar}_{\text{Poisson bracket}}\right)\right|}$$

$\rightarrow$ not both $\sigma_\mu(A), \sigma_\mu(B)$ can be made small if $\{A,B\} \neq 0$.

Proof : First, $\sigma_\mu(X) := \left(\mathbb{E}_\mu[X - \mathbb{E}_\mu(X)]^2\right)^{1/2}$ is "convex" in the sense that if you prove the inequality for "pure states", then it follows for all states.

Second, the set of states is convex, its extremal points are called "pure" states. $P^2 = P$ "projection"

For the class of examples $G(V)$ & $S(V)$ the pure states are $\boxed{\text{Projection operators } P_v,}$ with $P_v(w) = \langle v, w\rangle \cdot v$, for $v \in V$, and $\forall w \in V$.

Third, for a projection operator we'll have $\mathbb{E}_{P_v}(A) = \langle Av, v\rangle$.

Now, the proof : given A, B and $\mu = P_v$. we know

$$0 \leq \|(A + i\lambda B)v\|^2 = \underbrace{\langle A^2 v, v\rangle}_{c} + \lambda^2 \underbrace{\langle B^2 v, v\rangle}_{a} - i\lambda \overbrace{\langle (AB-BA)v, v\rangle \cdot \tfrac{\hbar}{\hbar}}^{b} \quad (\#)$$

Since the y-value of $ax^2 + bx + c$ minimizes at $c - \frac{b^2}{4a}$, $(\#)$ implies that

$$\langle A^2 v, v\rangle - \frac{\hbar^2\, \mathbb{E}_\mu(\{A,B\}_\hbar)^2}{4 \cdot \langle B^2 v, v\rangle} \geq 0 \iff \langle A^2 v, v\rangle \langle B^2 v, v\rangle \geq \frac{\hbar^2}{4} \langle \{A,B\}_\hbar v, v\rangle^2$$

Substitute A by $A - \mathbb{E}_\mu(A)$, same for B. $\Rightarrow$ lhs becomes $\sigma_\mu(A)\cdot\sigma_\mu(B)$ after taking $\sqrt{\ }$.

$\blacksquare$