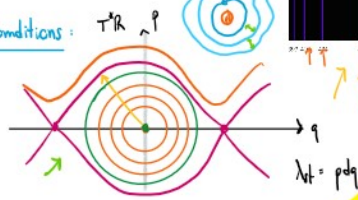
Lecture 11 : Matrix quantum mechanics (Heisenberg 1900–1925)

Hyd. ← Bohr

1. **Bohr-Sommerfeld quantization conditions:** $T^*\mathbb{R}$

Trying to select certain orbits. They are:

$$\boxed{\int_{orbit} \lambda_{st} \in \mathbb{Z}^{\geq 0} \cdot \hbar.}$$



$\lambda_{st} = p\,dq$

For the example $\lambda_{st} = p\,dq$, orbit is S^1_R:

$$\pi R^2 = \int_{|z|=R} p\,dq = n \cdot \hbar \Rightarrow$$ allowed orbits must have $\boxed{R^2 = \frac{n \cdot \hbar}{\pi}}$

modern: Bohr-Sommerfeld Lagrangian submanifold L.

is with $\mathbb{Z}$-periods. $\int_\gamma \lambda_{st}$, $\gamma \in H_1(L)$

2. Heisenberg's idea was to have observable

$$\hat{q} = (\hat{q}_{ij})$$ with $\hat{q}_{ij}$ = "$E_i - E_j$" difference in energy

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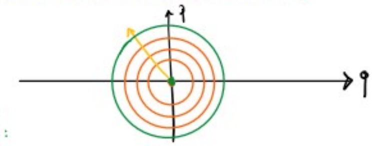
§ 1. Quantum Harmonic Oscillator : classical harmonic oscillator in $\mathbb{R}$

$$H(q,p) = \frac{1}{2}(p^2 + q^2) \quad \text{the energy}$$



Two observables are $q(t), p(t)$. At energy E orbit:

think of Fourier expansion

$q(t) = \sqrt{2E} \cdot \cos(t)$, $p(t) = \sqrt{2E} \cdot \sin(t)$ ⇝ $\dot{q} = \{q, H\}$, $\dot{p} = \{p, H\}$ Hamilt. eq^{ns}

Note that $a(t) = q(t) + ip(t) = \sqrt{2E} \cdot e^{it}$, $\bar{a}(t) = q(t) - ip(t)$ ⇝ $H = a \cdot \bar{a}$.

Since $a(t)$ only has frequency 1, the matrix (a_{ij}) will only have non-zero entries at $a_{i,i+1}$

$$a_{ij} = \begin{pmatrix} 0 & \sqrt{2E_1} & & \cdots \\ 0 & 0 & \sqrt{2E_2} & \cdots \\ 0 & 0 & 0 & \sqrt{2E_3} \\ \vdots & & & \end{pmatrix} \xrightarrow[q,p]{\text{recovers}} \hat{q}(0) = \sqrt{\frac{\hbar}{4\pi}} \begin{pmatrix} 0 & \sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \sqrt{2} & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \\ & & & & \ddots \end{pmatrix}$$

In summary, the observables q, p get quantized initially to

$$\hat{q}(0) = \sqrt{\frac{\hbar}{4\pi}} \cdot \begin{pmatrix} 0 & \sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \sqrt{2} & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \\ & & & & \ddots \end{pmatrix}, \quad \hat{p}(0) = \sqrt{\frac{\hbar}{4\pi}} \cdot \begin{pmatrix} 0 & -i\sqrt{1} & & \\ i\sqrt{1} & 0 & -i\sqrt{2} & \\ & i\sqrt{2} & 0 & -i\sqrt{3} \\ & & i\sqrt{3} & 0 \\ & & & & \ddots \end{pmatrix}$$

← hermitian operator $A^t = A$

How do $\hat{q}(0)$ and $\hat{p}(0)$ evolve? $\hat{q}(t), \hat{p}(t)$ are determined by Poisson bracket:

E.g. the evolution of $\hat{q}(t)$ depends on: $\boxed{\frac{d}{dt}\hat{q}(t) = \{\hat{q}(t), H\}}$ $\overset{?}{=} \frac{i}{\hbar}[A,B]$

The Hamiltonian H gets quantized:

$$\hat{H} = \begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & E_3 & \\ & & & \ddots \end{pmatrix}$$ ← diag. and eigenvalues are Energies allowed

$$\begin{pmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \cdots \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \cdots \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} = \left(\hat{q}H - H\hat{q}\right)_{ij}$$ is $\frac{i}{\hbar}(E_i - E_j) \cdot q_{ij}(t)$

In conclusion, $\hat{q}(t)' = \{\hat{q}(t), H\}_\hbar$ implies $\dot{q}_{ij}(t) = \frac{i}{\hbar}(E_i - E_j)\, q_{ij}(t)$

Hence $\boxed{\hat{q}_{ij}(t) = \hat{q}_{ij}(0) \cdot e^{\frac{i}{\hbar}(E_i - E_j)t}}$, similarly extract $\hat{p}(t)$. ∎

Note that $\{\hat{q}, \hat{p}\}_\hbar = i\hbar \cdot Id$, hence $\nexists$ a common eigenbasis.

So when we measure the observable $\hat{q}(t)$, we extract an eigenvalue of $\hat{q}$, λ_i, and right after the system is in the state μ given by $M = P_{v_{\lambda_i}}$, v_{λ_i} eigenvector for λ_i.

Since we cannot diagonalise $\hat{p}$ too, the expectation $\mathbb{E}_{P_{v_{\lambda_i}}}(\hat{p})$ we differed from observation. ∎