Lecture 12

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Lecture 12: Schrödinger's Equation: V $\mathbb{C}$-v.s., algebra of observables $\mathcal{H}(V)$

Initially $A(0)$, then $A(t)$ given by $\boxed{\dot{A}(t) = \{A(t), H\}_\hbar}$, states remain constant. $(\mu_t = \mu_0)$.

given "theory"

$\mathbb{E}_\mu(A) = tr(M \cdot A)$

We get $\boxed{A(t) := e^{-\frac{i}{\hbar}H\cdot t} \cdot A(0) \cdot e^{i/\hbar H \cdot t}}$, solves

$\rightarrow$ check: $\dot{A}(t) = \left(-\frac{i}{\hbar}\cdot H\right) A(t) + \frac{i}{\hbar} A(t)\cdot H = \frac{i}{\hbar}\left(AH - HA\right) = \{A, H\}_\hbar$.

Remark: (1) Classically, $H: T^*M \to \mathbb{R}$, then evolution of system is given by flow of X_H $\Big\} \to$ quantumly, the evolution is given by conjugation of
$\mathcal{U}(t) \equiv \mathcal{U}_t := e^{\frac{i}{\hbar}Ht}$ $\quad (A(t) = \mathcal{U}_{-t} A \,\mathcal{U}_t)$.

$\mathcal{U}$ gives the dynamics for the quantum system

(2) In the Schrödinger picture: $\dot{M}(t) = -\{M(t), H\}_\hbar$,
so $M(t) = \mathcal{U}(t)\, M(0) \cdot \mathcal{U}(-t)$.

Lemma: $\mathbb{E}_{\mu_t}(A) = \mathbb{E}_\mu(A(t))$. $\Rightarrow$ Heisenberg is equiv. to Schrödinger (exercise!)

Let's take Schrödinger's perspective and choose $\mu = \mu(0)$ a pure state. — projection operator P_v, where $v \in V$. (Ψ, $L^2(M,\mu)$)

By the Remark (2), a state will evolve according to

$\boxed{\mathcal{U}(-t)\cdot P_v \cdot \mathcal{U}(t) = P_{\mathcal{U}(-t)\cdot v}}$ $\rightsquigarrow$ all the info. is in $v_t := e^{-i/\hbar H\cdot t}\cdot v$. $\leftarrow$ how to find this? (it satisfies an eq^n.)

compute $P_v(A) = (Av, v)$

Since $\boxed{\frac{d}{dt} v_t = -\frac{i}{\hbar} H \cdot v_t}$ $\leftarrow$ $\boxed{\text{THIS THE SCHRÖDINGER EQ}^n}$, in addition this tells us the $P_v A = \frac{(Av, v)}{\|v\|^2}$

Stationary states: suppose $Hv = \lambda v$, i.e. v is eigenvector
then $v_t = e^{-i/\hbar H\cdot t} v = \underbrace{e^{-i/\hbar \lambda \cdot t}}\cdot v$ $\leftarrow$ this is a fct of t, v_t changes by $\mathbb{C}$-scalar
so $P_{v_t} = P_v$ is stationary.

Examples: (i) Free particle on a line: $\mathcal{H}(V)$ observ. with $V = L^2(\mathbb{R}, \mu)$. $\rightsquigarrow$ observables q, ∂_q

Then $H(q,p) = \frac{p^2}{2m}$ quantize to $H = \frac{-\hbar^2}{2m}\partial_q^2$, i.e. $H\Psi = \frac{-\hbar^2}{2m}\cdot \Psi''(q)$, $\Psi = \Psi(q) \in L^2(\mathbb{R})$.

The Schrödinger eq^n for $\Psi(q,t)$: $\partial_t \Psi = \frac{-i}{\hbar}\cdot\frac{-\hbar^2}{2m}\cdot\partial_q^2 = \frac{i\hbar}{2m}\partial_q^2\Psi(q,t)$.

unbounded op.!

$$\boxed{\partial_t \Psi = i\cdot\frac{\hbar}{2m}\cdot\partial_q^2\Psi}$$

(ii) In general, a free particle on M will have $V = L^2(M, \mu)$, and the Hamiltonian ($\leftarrow$ observable position?)
is $H = C\cdot\Delta$
$\hookrightarrow$ Laplacian which depends on metric (M, g). e.g. $M = S^2$ round metric

In particular, eigenstates for the energy (stationary states) are eigenfcts of Δ.

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§2. Hilbert spaces: In general, our algebra of observ. will be $\mathcal{H}(V)$ with V Hilbert space.

Def^n: An operator $M: V \longrightarrow V$ is nuclear if $\exists$ basis (for any basis) (e_n) e.g. $L^2(X, \mathbb{C})$ ($\rightarrow$ think of as density fct for a state)

if $\mathrm{Tr}(M) := \sum_{n\geq 1} \langle M\cdot e_n, e_n\rangle_V$ is convergent. (see "trace-class")

L^2-product: e.g. $\int_X e_n \bar{e}_n \, d\mu_n$

Def^n: A state $\mu: \mathcal{H}(V)_{bd} \longrightarrow$ prob meas. on $\mathbb{R}$ defined by the expectations (for well-def. on bounded)

A bounded op.: $\mathbb{E}_\mu(A) = \mathrm{Tr}(\underbrace{M\cdot A}_{\text{is finite}})$, w/ M self-adj., non-neg., nuclear & $tr(M) = 1$.