

Lecture 13: Quantum Particle in  $S^1$  ~ compact,  $L^2(S^1, \mathbb{C})$  easier than  $L^2(\mathbb{R}, \mathbb{C})$

Motivation: study  $C_6H_6$  (benzene)  spectrum is discrete:  $\int_{S^1}$  better  $\int_{\mathbb{R}}$

Std. questions: given a state  $\Psi \in L^2(S^1, \mathbb{C})$  on Hilbert space, } given  $\hat{H}$  the energy observable

(i) what energies are possible and their prob.?

study  $E_\Psi, \delta_\Psi$ .

(ii) Evolution of the system?  $\rightarrow$  stationary states? By Monday, these are eigenvects of energy operator.

If  $\Psi$  at  $t=0$  is given, how does it evolve? } solve Schrödinger PDE.

$$i\hbar \partial_t \Psi = \hat{H} \Psi$$

Evaluation is given by Schrödinger:  $i\hbar \partial_t \Psi = \hat{H} \Psi$ , by sep. of variables

the states  $\Psi_n(\theta, 0)$  evolve on  $\Psi_n(\theta, t) = e^{-\frac{i}{\hbar} E_n t} \cdot \Psi_n(\theta, 0)$  ~ 'wave' mech.

Example: If the system starts at  $P_{\Psi_n}$ , what can we say?

The expected value  $E_{P_{\Psi_n}}(\hat{H}) = \langle \Psi_n, \hat{H} \Psi_n \rangle_{L^2} = E_n$ .

What is the std. deviation?  $\delta_{P_{\Psi_n}}(\hat{H})^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 = \langle \Psi_n, \hat{H}^2 \Psi_n \rangle - E_n^2 = 0$ .

What about momentum?  $E_{P_{\Psi_n}}(\hat{p}) = \langle \Psi_n, \hat{p} \Psi_n \rangle = \hbar n$ . (check  $\delta(\hat{p}) = 0$ )  
 $\hat{p} = -i\hbar \partial_\theta$ .  $\hat{p} \Psi_n = \hbar n \cdot \Psi_n$ , eigenstate also for  $\hat{p}$ ! (note  $\langle \hat{H}, \hat{p} \rangle = 0$ )

In our problem, the Hilbert space is  $L^2(S^1, \mathbb{C})$ . (Assume radius  $R$  of  $S^1$  is  $R=1$ )  $\frac{1}{R^2} \cdot \partial_\theta^2$   
 The energy is  $\hat{H} = \frac{-\hbar^2}{2m} \partial_\theta^2 \rightarrow$  classically  $H = \frac{1}{2m} p^2 + p$  quantized to  $-i\hbar \partial_\theta$ .

What are the stationary states? We need to solve  $\hat{H} \Psi = E \cdot \Psi$ , for some  $E \in \mathbb{R}^+$ .

i.e.  $\partial_\theta^2 \Psi(\theta) = \frac{-2mE}{\hbar^2} \Psi(\theta) \rightarrow \Psi(\theta) = C \cdot e^{\pm i\theta \cdot \frac{\sqrt{2mE}}{\hbar}}$   $\leftarrow \Psi$  will become  $\Psi(\theta, t)$   
 gen. sol. superposition

Note that  $\theta \in S^1$ , so  $\Psi(\theta) = \Psi(\theta + 2\pi)$ , hence we get

$\frac{\sqrt{2mE}}{\hbar} = n, n \in \mathbb{Z} \Rightarrow E_n = \frac{\hbar^2 n^2}{2m}$  energies associated to  
 $\Psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta} \rightarrow \|\Psi_n\| = 1$  implies  $C = \frac{1}{\sqrt{2\pi}}$   
 the stat. states are  $P_{\Psi_n}$ , with  $\Psi_n$  as defined.

What about position? In the state  $\Psi_n$ ,

$\hat{q}_1 = \cos \theta, \hat{q}_2 = \sin \theta$   
 $E_{P_{\Psi_n}}(\hat{q}_1) = \langle \Psi_n, \hat{q}_1 \Psi_n \rangle_{L^2} = \int_0^{2\pi} \Psi_n \cdot \cos \theta \cdot \overline{\Psi_n} d\theta = 0$ .

Note that  $\delta_{P_{\Psi_n}}^2(\hat{q}_1) = \langle \Psi_n, \hat{q}_1^2 \Psi_n \rangle - 0 = \int_0^{2\pi} \cos^2 \theta d\theta = \pi$ , not zero!

Mixed states: Suppose the system is initially in, e.g.  $\Psi := \sqrt{2} \cdot \left( \frac{1}{\sqrt{3}} \sin \theta + \sqrt{\frac{2}{3}} \cos(3\theta) \right)$  we can expand in terms of  $\Psi_n$ , we get superposition of  $\Psi_1, \Psi_{-1}, \Psi_3, \Psi_{-3}$ .

Here, the momentum  $\hat{p}$  will acquire 2 possible values for  $\hat{p}$ :  
 each of them  $\frac{1}{6} \rightarrow \pm \hbar$  each of them  $\frac{1}{3} \rightarrow \pm 3\hbar \rightarrow \frac{1}{6} \text{ in } \Psi_1 + \frac{1}{6} \text{ in } \Psi_{-1} + \frac{1}{3} \text{ in } \Psi_3 + \frac{1}{3} \text{ in } \Psi_{-3}$