Lecture 14

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Lecture 14 : Canonical Quantization

one of many possible

Classically, a phase space $M = T^*Q$ ($Q = S^1$) $\xrightarrow{\text{quantization}}$ $V = L^2(Q, \mathbb{C})$ e.g. — Hilbert space

$G(M) := C^\infty(T^*Q) \xrightarrow[\text{observable}]{\text{quantize}} G(V) :=$ hermit. operators. — typically can be unbounded, ask defined in dense subset

e.g. $Q = \mathbb{R}_q$, q, p

$f \longmapsto A_f$? $q\cdot$, $-i\hbar\partial_q$

$p \longmapsto M_p$?

Two points of focus : (1) Build this algebra morphism s.t. $\boxed{\lim_{\hbar \to 0} A_{\{f,g\}} = \lim_{\hbar\to 0} \{A_f, A_g\}_\hbar}$.

How to build this ? Decide where q, p go first,

then study where $f(q,p)$ goes.

$(f(q,p) = q\cdot p \rightsquigarrow A_f = q\partial_q\ ?)$

(2) Uniqueness : morally, q & p can "only" be sent to $q\cdot$ and $-i\hbar\partial_q$. ✓

It'll be STONE-VON NEUMANN theorem

§ 1. Stone-Von Neumann Thm. : $Q = \mathbb{R}$

First, we consider the ansatz $A_q\Psi := q\cdot\Psi$, $A_p\Psi := -i\hbar\partial_q\Psi$. — self-adjoint hermitian

not defined on all $L^2(\mathbb{R}, \mathbb{C})$ → defined in dense subset of L^2. } $\mathfrak{g}$ level

Q : Is this unique ? No, but the counter-ex. all come from using unboundedness.

In order to get to sound ground, we $\boxed{\text{EXPONENTIATE}}$: e^{iA_q}, e^{iA_p} are unitary and bounded.

The previous rel^n was $\{A_q, A_p\} = 1$. Now, the exponential version is — G Lie gp.

this is our new requirement → $\boxed{e^{iA_q\cdot t}\cdot e^{iA_p\cdot s} = e^{ist}\cdot e^{iA_p\cdot s}\cdot e^{iA_q t}}$ ⇝ now all is bounded and we are down to Lie group.

Thm : (Stone-Von Neumann) For each $\hbar \in \mathbb{R}^\times$, $\exists$ an irreducible unitary representation

$\rho_\hbar : G_{Heis} \longrightarrow \mathcal{U}(L^2(\mathbb{R}^n, \mathbb{C}))$ given by — tells you how A_q, A_p act

$$\boxed{\rho_\hbar(g_{\alpha,\beta,\gamma})\cdot\Psi(q) := e^{i(\gamma q + \beta\cdot\hbar)}\cdot\Psi(q + \alpha\cdot\hbar)},$$

⇝ moral is that e^{iA_q}, e^{iA_p} act uniquely once $\hbar$ is given.

such that any unitary rep^n is equivalent to one such $\rho_\hbar$. ∎

In thm : $G_{Heis} := \left\{ \begin{pmatrix} 1 & \vec{\alpha} & \beta \\ & \mathrm{Id}_n & \vec{\gamma} \\ & & 1 \end{pmatrix} \in \mathrm{End}(\mathbb{R}^{n+2}),\ \vec{\alpha}, \vec{\gamma} \in \mathbb{R}^n,\ \beta \in \mathbb{R} \right\}$, Lie group.

Really, study $\mathfrak{g}_{heis} = \left\{ \begin{pmatrix} 0 & \vec{a} & b \\ & 0 & \vec{c} \\ & & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{R}^{n+2}) \right\}$ generated by $(n=1)$ $e_{12} = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & \\ & & 0 \end{pmatrix}$ "q", $\begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{pmatrix} = e_{13}$ $\hbar\cdot\mathrm{Id}$, $\begin{pmatrix} 0 & 0 & 0 \\ & & 1 \\ & & 0 \end{pmatrix} = e_{23}$ "p"

$[e_{12}, e_{23}] = e_{13}$ $(e_{ij}\, e_{kl} = \delta_{jk} e_{il})$