Lecture 15

Tuesday, September 29, 2020 12:05 PM

Lecture 15 : Weyl's quantization and Integral operators

For simplicity $M = \mathbb{R}_q$ and $(q,p) \in T^*\mathbb{R}_q$.

The goal is quantizing $\underbrace{G(T^*M)}_{\text{classical obs}}$ to $\underbrace{\mathcal{H}(L^2(M,\mathbb{C}))}_{\text{quantum obs.}}$.

→ **Step 1** : quantize position $q \in G(T^*\mathbb{R})$ and momentum $p \in G(T^*\mathbb{R})$ to operators $\hat{q}$ and $\hat{p}$. As discussed on Friday, we instead focus of $e^{it\hat{q}}$ and $e^{it\hat{p}}$. (unitary & bounded)

(i) How do $e^{it\hat{q}}$ and $e^{it\hat{p}}$ act ?

(ii) How unique is this representation ?

→ Answers given by $\boxed{\text{STONE-VON NEUMANN}}$ thm.

→ **Step 2** : quantize an arbitrary observable $f = f(q,p) \in G(T^*\mathbb{R}_q)$.

this is today's topic : understanding how, why and properties.

§ 1. Quantizing observables

: we know $e^{it\hat{q}}, e^{it\hat{p}}$. Then the inverse Fourier formula :

$$f(q,p) = \int_{\mathbb{R}_s}\int_{\mathbb{R}_t} \mathcal{F}(f)(s,t)\cdot e^{i(sq+tp)}\, ds\,dt.$$

quantize : express this with $e^{it\hat{p}}, e^{is\hat{q}}$

$\hat{p} = i\hbar\partial_q$

use $e^{i(s\hat{q}+t\hat{p})} \overset{\text{BCH}}{=} e^{is\hat{q}} e^{it\hat{p}} \cdot e^{i\hbar st/2}$, and $e^{it\hat{p}}\Psi(q) = \Psi(q+t\hbar)$:

quantize !

$$\boxed{A_f\Psi(q) := \int_{\mathbb{R}_s}\int_{\mathbb{R}_t} \mathcal{F}(f)(s,t)\cdot e^{i\hbar ts/2}\cdot e^{itq}\cdot\Psi(q+\hbar t)\, ds\,dt}$$

"Weyl's quantization"

4 integrals

does this quantize as we want ? $\lim_{\hbar\to 0}\{A_f, A_g\} = \lim_{\hbar\to 0} A_{\{f,g\}}$

regularity ? Is $f\in C^\infty$ enough for boundedness ? No ⇝ distributional !

Thm : Consider the quantization A_f, then :

(1) A_f is an integral operator with kernel

$$K(q,y) = \int_{\mathbb{R}_q} \underbrace{f\left(\frac{q+y}{2}, w\right)}_{\text{symbol}} e^{-iw\left(\frac{y-q}{\hbar}\right)}\, dw,$$

depend on f

i.e.

$$\boxed{A_f\cdot\Psi(q) = \int_{\mathbb{R}_q} K(q,y)\,\Psi(y)\,dy.}$$

(2) $\lim_{\hbar\to 0}\{A_f, A_g\} = \lim_{\hbar\to 0} A_{\{f,g\}_\hbar}$ (see Dolgachev's Lect. 6) ∎

Proof : we have

$$A_f\Psi(q) = \int_{\mathbb{R}_s}\int_{\mathbb{R}_t}\left(\int_{\mathbb{R}_z}\int_{\mathbb{R}_w} f(z,w)\,e^{-i(zs+wt)}\,dz\,dw\right)e^{i\hbar st/2}\,e^{isq}\,\Psi(\underbrace{q+\hbar t}_{y})\,ds\,dt$$

$$= \hbar^{-1}\int_{\mathbb{R}_s}\int_{\mathbb{R}_y}\left(\int_{\mathbb{R}_z}\int_{\mathbb{R}_w} f(z,w)\cdot e^{-i\left(zs+w\left(\frac{y-q}{\hbar}\right)\right)}\,dz\,dw\right)e^{\frac{i\hbar s(y-q)}{2\hbar}}\,e^{isq}\,\Psi(y)\,ds\,dy =$$

$$= \hbar^{-1}\int_{\mathbb{R}_s}\int_{\mathbb{R}_y}\int_{\mathbb{R}_z}\int_{\mathbb{R}_w} f(z,w)\cdot e^{-iw\left(\frac{y-q}{\hbar}\right)}\cdot\underbrace{e^{-izs}\cdot e^{\frac{is(y-q)}{2}}\cdot e^{isq}}_{e^{is\left(\frac{y+q}{2}-z\right)}}\cdot\Psi(y)\,dz\,dw\,ds\,dt$$

$$= \hbar^{-1}\int_{\mathbb{R}_y}\int_{\mathbb{R}_z}\int_{\mathbb{R}_w} f(z,w)\,e^{-iw\left(\frac{y-q}{\hbar}\right)}\cdot\delta\left(z-\left(\frac{y+q}{2}\right)\right)\Psi(y)\,dz\,dw\,dt$$

which equals the required expression. ∎

§ 2. Properties and examples :

(i) Behaviour under symplectomorphism : $f\in C^\infty(T^*M) \rightsquigarrow A_f$

if $\phi \in \mathrm{Symp}(T^*M)$ then $f\circ\phi$ quantizes to $A_{f\circ\phi} = \Phi\circ A_f\circ\Phi^*$.

thm : $\exists\,\Phi$ operator $\Phi: S' \to S'$

"canonical transformation"

$Sp(\mathbb{R}^{2n}, \omega_{st})$: in particular $\tau\in \mathrm{Diff}(M)$ gives $\phi_\tau \in \mathrm{Symp}(T^*M)$

(ii) Calderon - Vaillancourt Thm : if $f\in C^\infty(T^*\mathbb{R}^n)$ and $\|\partial_q^i\partial_p^j f\| \leq (1+|q|+|p|)^\alpha\cdot C_{i,j}$.

then A_f is bounded. (the key step $\|A_f\| \leq C\cdot\|f\|_\infty^?$.) (see Folland.)

a fct. localized at (q_0,p_0)

Example : a wave packet $\gamma_{(q_0,p_0)} \in C^\infty(T^*\mathbb{R}^n)$ transforms nicely :

$$A_f\cdot\gamma_{(q_0,p_0)} = \Big(\underset{\text{in deg } k}{\text{Taylor series of } f}\Big)_{\text{at }(q_0,p_0)}(\hat{q},\hat{p})\cdot\gamma_{(q_0,p_0)} + O(\hbar^{k\text{-const}}).$$

$(\hat{q}-q_0)(\hat{p}-p_0)$

e.g. $\gamma_{(q_0,p_0)} = e^{-\frac{(q-q_0)^2}{2}}\cdot e^{ip_0 q/\hbar}$