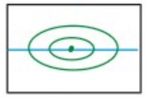
Lecture 16

Tuesday, September 29, 2020 12:05 PM

Lecture 16 : THE HARMONIC OSCILLATOR (via "ladder method"; i.e. rep^n thy.)

The classical Hamiltonian: $H(q,p) = \frac{p^2}{2m} + \frac{\omega^2 m}{2} q^2$, suppose $m=1$.



Canonical quantization: $\hat H = \frac{(-i\hbar\partial_q)^2}{2m} + \frac{\omega^2 m}{2}\hat q^2$, acts on $L^2(\mathbb{R},\mathbb{C})$.

questions: stationary states, expected values, possible energies? some info on $\hat q, \hat p$

The main idea is to factorize $\hat H$ using $a = \frac{1}{\sqrt{2\omega}}(\omega\hat q - i\hat p)$ and $a^* = \frac{1}{\sqrt{2\omega}}(\omega\hat q + i\hat p)$.

First, note $\omega\cdot aa^* = \omega\cdot\frac{1}{\sqrt{2\omega}}(\omega\hat q - i\hat p)\frac{1}{\sqrt{2\omega}}(\omega\hat q + i\hat p) \overset{[\hat q,\hat p]=i\hbar}{=} \hat H + \frac{\hbar\omega}{2}$

Similarly $\omega\cdot a^*a = \hat H - \frac{\hbar\omega}{2}$ ←

$$\Rightarrow \boxed{[a,a^*] = \hbar,\quad [H,a] = -\hbar\omega\cdot a,\quad [H,a^*] = \hbar\omega a^*}$$

→ Lie algebra $\mathfrak{h}^{ext}$ generated by $1, a, a^*, H$ (extension)

↳ note that $1, a, a^*$ this Heisenberg Lie algebra $\mathfrak{h}$.

This is now the algebraic problem of representing $\mathfrak{h}^{ext}$ in some space. ($L^2(\mathbb{R},\mathbb{C})$ for us.)

Q: $\exists$ ft. dim'l V v.s. such that $\mathfrak{h}^{ext}$ acts faithfully and irreducibly? "commutative enough"

A: No. We need ∞-dim'l V's, such as $L^2(\mathbb{R},\mathbb{C})$.

↳ solvable Lie alg. only have 1-dim'l irrep.

Start with H and $\Psi \in L^2(\mathbb{R},\mathbb{C})$ an (normalised) eigenvector. Then

for all irreps → $\lambda\cdot\hbar\|\Psi\|^2 = \langle\Psi, H\Psi\rangle = \langle\Psi, \omega a^*a\Psi + \frac{\hbar\omega}{2}\Psi\rangle = \omega\langle\Psi, a^*a\Psi\rangle + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}$

$H\Psi = \lambda\Psi$ $\qquad \langle a\Psi, a\Psi\rangle \geq 0$

In fact, the eigenvalue is $\lambda = \frac{\hbar\omega}{2}$ if $\langle a\Psi, a\Psi\rangle = 0$ i.e. $a\Psi = 0$.

FUNDAMENTAL COMPUTATION: since $[H,a] = -\hbar\omega a \Rightarrow Ha\Psi - a\underbrace{H\Psi}_{\lambda\Psi} = -\hbar\omega a\Psi$

so, if Ψ is H-eigenvector, then $H(a\Psi) = (-\hbar\omega + \lambda)a\Psi$. So $a\Psi$ is H-eig. with $\lambda - \hbar\omega$.

Prop: Let Ψ be an H-eig. w/ eigenvalue λ. Then

(i) $a\Psi$ is H-eig with eigenvalue $\lambda - \hbar\omega$

(ii) $a^*\Psi$ is H-eig with eigenvalue $\lambda + \hbar\omega$ ∎

⇝ start with Ψ being lowest eigv. $H\Psi = \frac{\omega\hbar}{2}\Psi$, then $a^*\Psi$ have H-eigenvalues $\lambda_n = \left(\frac{2n+1}{2}\right)\hbar\omega$, $n=0,1,2\dots$

$\Rightarrow$ possible energies $E_n = (2n+1)\cdot\frac{\hbar\omega}{2}$, $n=0,1,2\dots$ ← smallest energy $\neq 0$!

How to get all stationary states? (1) Find $\Psi = \Psi_0$ the one w/ lowest energy $\frac{\hbar\omega}{2}$, then $\Psi_n = (a^*)^n\Psi_0$ (and normalize)

(2) Ψ_0 is just given by $H\Psi_0 = \frac{\hbar\omega}{2}\Psi_0$, but better use $a\Psi_0 = 0$. $\Rightarrow$ Ψ_0 solves $\boxed{(\omega\hat q + \hbar\partial_q)\Psi_0 = 0}$

so $\Psi_0 = C\cdot e^{-\frac{\omega q^2}{2\hbar}}$, $C = \sqrt[4]{\frac{\omega}{\hbar\pi}}$ ∎

Summary of Quantum Harm. Osc.:

$\hat H = \frac{\hat p^2}{2} + \frac{\omega^2\hat q^2}{2}$, $a = \frac{1}{\sqrt{2\omega}}(\omega\hat q - i\hat p)$, a^* adjoint.

Note that $\hat q = \frac{a + a^*}{\sqrt{2\omega}}$ and $\hat p = \frac{\sqrt{2\omega}(a^* - a)}{2i}$.

↳ extended Heisenberg $[a,a^*] = \hbar$, $[H,a] = -\hbar\omega a$, $[H,a^*] = \hbar\omega a^*$

The eigenvalues of H are (lowest energy is $E_0 = \hbar\omega/2$) ($\langle\Psi,H\Psi\rangle \geq \frac{\hbar\omega}{2}$ $\forall\Psi$ eigen.)

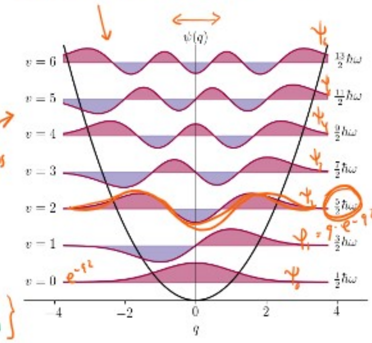
$E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0,1,2\dots$ stationary states

with eigenvectors $\boxed{\Psi_n := \frac{1}{\sqrt{\hbar^n n!}}(a^*)^n\cdot\Psi_0}$ ↳ orthonormal basis

where Ψ_0 solves:

$a\Psi_0 = 0$ (ode) $\Rightarrow$ $\boxed{\Psi_0(q) = \left(\frac{\omega}{\hbar\pi}\right)^{1/4} e^{-\omega q^2/2\hbar}}$

$a^*a\,\Psi_n$, $n\Psi$, $a^*a\,\phi = \lambda\phi$, $(a\phi)$, $\lambda - 1$



{ useful computation: $a^*\Psi_n = \sqrt{(n+1)\hbar}\cdot\Psi_{n+1}$ & $a\Psi_n = \sqrt{n\hbar}\cdot\Psi_{n-1}$ }

Problem 1: In each stationary state Ψ_n, compute $\mathbb{E}_{\Psi_n}(\hat{q})$, $\mathbb{E}_{\Psi_n}(\hat{p})$, $\sigma_{\Psi_n}(\hat{q})$ and $\sigma_{\Psi_n}(\hat{p})$.

Sol^n: $\mathbb{E}_{\Psi_n}(\hat{q}) = \langle \Psi_n, \hat{q}\Psi_n \rangle_{L^2} = \langle \Psi_n, \frac{1}{\sqrt{2\omega}}(a+a^*)\Psi_n \rangle = 0.$

$a\Psi_n = \sqrt{(n-1)\hbar}\cdot\Psi_{n-1}$

$a^*\Psi_n = \sqrt{(n+1)\hbar}\,\Psi_{n+1}$

Similarly, $\mathbb{E}_{\Psi_n}(\hat{p}) = 0$. So particle in Ψ_n is at rest at origin.

$\langle \Psi_n, n\hbar\Psi_n \rangle$

Since $\mathbb{E}_{\Psi_n}(\hat{q}^2) = \langle \Psi_n, \hat{q}^2\Psi_n \rangle = \frac{1}{2\omega}\Big(\overset{0}{\langle \Psi_n, a^2\Psi_n \rangle} + \overset{0}{\langle \Psi_n, a^{*2}\Psi_n \rangle} + \underbrace{\langle \Psi_n, aa^*\Psi_n \rangle}_{\langle \Psi_n, ((n+1)\hbar)\Psi_n \rangle} + \langle \Psi_n, a^*a\Psi_n \rangle \Big)$

$= \frac{1}{2\omega}\cdot(2n+1)\hbar$, we get $\sigma_{\Psi_n}(\hat{q}) = \left(\frac{2n+1}{2}\,\frac{\hbar}{\omega}\right)^{1/2}$.

Similarly, $\mathbb{E}_{\Psi_n}(\hat{p}^2) = \hbar\omega\left(\frac{2n+1}{2}\right)$

$$\boxed{\sigma_{\Psi_0}(\hat{q})\cdot\sigma_{\Psi_0}(\hat{p}) = \frac{\hbar}{2}}$$

(Heisenberg's ineq. is sharp!)

Problem 2: In the state Ψ_n, compute the potential and kinetic energies.

Sol^n: In Ψ_n we have energy $E_n = \left(\frac{2n+1}{2}\right)\hbar\omega$. How does it break up in $K + U$?

Since $K = \frac{p^2}{2}$, $\mathbb{E}_{\Psi_n}\left(\frac{p^2}{2}\right) = \left(\frac{2n+1}{2}\right)\cdot\frac{\hbar\omega}{2}$,

also $\mathbb{E}_{\Psi_n}\left(\frac{q^2\omega^2}{2}\right) = \left(\frac{2n+1}{2}\right)\cdot\frac{\hbar\omega}{2}$

50% kinetic
50% potential
(expectedly!)

How about superposed states? $\rightsquigarrow$ (must $\mathbb{E}_{\mu}(\hat{q}) = 0$?)

Prob. 3: what is the expected position of a particle in the state $\boxed{\varphi := \frac{1}{\sqrt{2}}(\Psi_0 + \Psi_1)}$?

Sol^n: $\langle \varphi, \hat{q}\varphi \rangle = \frac{1}{2}\Big(\overset{0}{\langle \Psi_0, \hat{q}\Psi_0 \rangle} + \overset{a}{\langle \Psi_0, \hat{q}\Psi_1 \rangle} + \overset{a^*}{\langle \Psi_1, \hat{q}\Psi_0 \rangle} + \underbrace{\langle \Psi_1, \hat{q}\Psi_1 \rangle}_{0} \Big)$

$= \frac{1}{2}\Big(\langle \Psi_0, \frac{1}{\sqrt{2\omega}}\underbrace{a\Psi_1}_{\Psi_0\hbar^{1/2}} \rangle + \langle \Psi_1, \frac{1}{\sqrt{2\omega}}\underbrace{a^*\Psi_0}_{\Psi_1\hbar^{1/2}} \rangle \Big) = \left(\frac{\hbar}{2\omega}\right)^{1/2}$

$q = (a+a^*)\frac{1}{\sqrt{2\omega}}$

Prob. 4: what is the expected momentum of a particle at time $t = t_0$ if at $t=0$ the system is at the state $\boxed{\varphi := \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)}$?

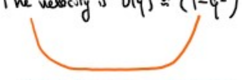
Sol^n: Need to understand evolution for φ: $\varphi(q,t) = e^{-iHt/\hbar}\varphi = \frac{1}{\sqrt{2}}\left(e^{-i\frac{3}{2}\hbar\omega t/\hbar}\cdot\Psi_1 + e^{-i\frac{5}{2}\hbar\omega t/\hbar}\cdot\Psi_2\right)$

hence $\langle \varphi(q,t), \hat{p}\,\varphi(q,t) \rangle = -(\hbar\omega)^{1/2}\sin(\omega t)$

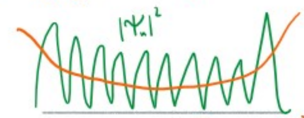
(i) **Relation between classical & quantum**:

What is the classical prob. dist Ψ_{class}?

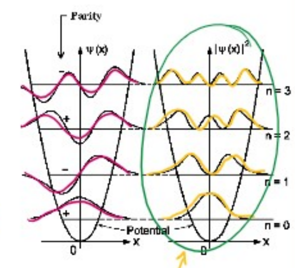
The velocity is $v(q) \approx (1-q^2)^{1/2}$



$|\Psi_n|^2$ for $n \to \infty$



$E_n = \frac{\hbar\omega}{2}\cdot(2n+1)$, $\hbar \to 0$, $n \to \infty$



probability distrib.

(ii) **Heisenberg's Picture revisited**:

Two Fridays ago used "matrix mech", write $a, a^*, \hat{q}, \hat{p}$ in the Ψ_n eigenbasis:

$$a = \begin{pmatrix} 0 & 0 & 0 & \\ \sqrt{1} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \\ & 0 & \sqrt{3} & \\ & & 0 & \\ & & \vdots & \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 & \sqrt{1} & 0 & \\ 0 & 0 & \sqrt{2} & \\ 0 & 0 & 0 & \sqrt{3} \\ & & & \ddots \end{pmatrix}$$

$\rightsquigarrow$ recover matrices for $\hat{q}$ & $\hat{p}$!