Lecture 17

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Lecture 17 : Central Potentials (using representations of $SO(3)$ in physics) $(x,y,z) \sim (r,\theta,\varphi)$ in $\mathbb{R}^3$

Let us consider a quantum particle in $\mathbb{R}^3$ with potential energy $V = V(r)$, $r^2 = x^2+y^2+z^2$.

Thm: The stationary states $\Psi(r,\theta,\varphi) \in L^2(\mathbb{R}^3)$ of a system $\boxed{H = \frac{\hat{p}^2}{2} + V(r)}$, if non-degenerate*, can be written as

indep of r

$$\boxed{\Psi(r,\theta,\varphi) = \underbrace{\frac{h(r)}{r}}_{r\text{ piece}} \cdot \sum_{m=-k}^{k} c_m Y_{km}(\theta,\phi)}$$

labeled by $k \in \mathbb{N}$ and $|m| \leq k$.

(i) $h(r)$ solves $\boxed{-\frac{\hbar^2}{2} h''(r) + \frac{\hbar(k+1)k}{2r^2} \cdot h(r) + V(r) \cdot h(r) = E \cdot h(r)}$ ← ode in r

(ii) $Y_{km}(\theta,\phi)$ are the spherical harmonics : $Y_{km}(\phi,\theta) = \frac{1}{\sqrt{2\pi}} \cdot e^{im\phi} \, P_k^m(\cos\theta)$

$L_3 \sim \partial_\phi$ — Legendre polynomial — from L_-

* Eigenvalue is non-deg.

§ 1. The symmetries of central systems : q, p and angular momentum $q \times p$

Write $\ell = q \times p = (\underbrace{q_2p_3 - p_2q_3}_{\ell_1}, \underbrace{q_3p_1 - p_3q_1}_{\ell_2}, \underbrace{q_1p_2 - q_2p_1}_{\ell_3})$ quantizes canonically $\hat{\ell} = (L_1, L_2, L_3)$.

We can compute $[L_1, L_2] = iL_3$, $[L_2, L_3] = iL_1$, $[L_3, L_1] = iL_2$,

$[L_i, H] = 0$ because H central. → what do they generate? Lie alg. gen. by L_1, L_2, L_3 + H piece.

Lemma: L_1, L_2, L_3 generate $so(3)$. ($[e_i, e_j] = e_k$, $e_j = iL_j$)

Key question: How does $L^2(\mathbb{R}^3)$ decompose under $SO(3)$ symmetries ?

(i) Peter-Weyl thm : any rep^n of $SO(3)$ is a direct sum of FINITE IRREDUCIBLE rep^n.

(ii) Irreps for $SO(3)$ are labeled by $k \in \mathbb{N}$: exactly 1 in dim^n $2k+1$. ($SO(3) \curvearrowright \mathbb{C}^{2k+1}$ irred.)

§ 2. Irreps for $SO(3)$: consider the space of harmonic polynomial with homog. degree k :

$SO(3) \circlearrowright \mathcal{H}^k := \{ P \in \mathbb{C}[x,y,z] : \underbrace{P \text{ homog. of deg. } k}_{(k+1)(k+2)/2} \text{ and } \Delta_{\mathbb{R}^3} P = 0 \}$ ← Ex: dim $2k+1$.

(because of homog. + "Δ metric") — describe the irrep.

Lemma: This is an irreducible rep. ← how do L_1, L_2, L_3 act ?

Remember the FUNDAMENTAL COMPUTATION : $\boxed{\left.\begin{array}{l}\Psi \text{ eigenvect for } M \text{ of eig. } \lambda \\ [M, A] = \alpha \cdot A\end{array}\right\} \Rightarrow A\Psi \text{ is then an eigenvect of } M \text{ with eigenvalue } \lambda + \alpha.}$

Consider $L_\pm = L_1 \pm i L_2$

then $\langle L_1, L_2, L_3 \rangle = \langle L_+, L_-, L_3 \rangle$.

$\left.\begin{array}{l}[L_3, L_+] = L_+ \\ [L_3, L_-] = L_-\end{array}\right\} \Rightarrow$ all needed to do is understanding L_3-eigenspaces

→ Find the "highest weight" : solve $L_+\Psi = 0$

$0 \xleftarrow{L_-} Y_{k,-k} \xleftarrow{L_-} \cdots \xleftarrow{L_-} Y_{k,k-1} \xleftarrow{L_-} Y_{k,k-1} \xleftarrow{L_-} \Psi = Y_{kk} \xrightarrow{L_+} 0$

→ Find the rest applying : $L_-^n \cdot \Psi$

§ 3. Spherical harmonics : by def^n these are restriction of $\mathcal{H}^k$, for some k, to S^2.

In $(\theta,\phi) \in S^2$ we write $L_3 = -i\partial_\phi$, $L_\pm = e^{\pm i\phi}(\pm\partial_\theta + i \cot\theta \cdot \partial_\phi)$

Hence we need to solve : $L_+\Psi = 0 \iff \partial_\theta \Psi + i \cot\theta \cdot \partial_\phi \Psi = 0$

remember Ψ is eig. of L_3 : $\boxed{-\partial_\phi \Psi = k \cdot \Psi}$, $k \in \mathbb{N}$ so $\Psi_k = e^{ik\phi} \cdot \chi_k(\theta)$

$\iff \boxed{\partial_\theta \chi_k(\theta) - k \cot\theta \cdot \chi_k(\theta) = 0}$

This is solved by $\chi_k(\theta) = \sin^k\theta$. ⇒ highest weight eigenvect. is $\boxed{Y_{kk}(\phi,\theta) = e^{ik\phi} \cdot \sin^k\theta}$.

Then apply L_- to get $\underbrace{Y_{km}}_{|m| \leq k} \sim C \cdot e^{im\phi} \cdot \text{Poly}(\sin\theta)$

found by applying L_- : labeled by $P_{mk}(z)$: legendre poly.