

Lecture 17: Central Potentials (using representations of $SO(3)$ in physics) $(x,y,z) \sim (r,\theta,\phi)$ in \mathbb{R}^3

Let us consider a quantum particle in \mathbb{R}^3 with potential energy $V = V(r)$, $r^2 = x^2 + y^2 + z^2$.

Thm: The stationary states $\Psi(r,\theta,\phi) \in L^2(\mathbb{R}^3)$ of a system $H = \frac{\hat{p}^2}{2} + V(r)$,

if non-degenerate, can be written as

$$\Psi(r,\theta,\phi) = \underbrace{\left(\frac{h(r)}{r}\right)}_{r \text{ piece}} \cdot \sum_{m=-k}^k c_m \underbrace{Y_{km}(\theta,\phi)}_{\text{indep of } r}$$

labeled by $(k \in \mathbb{N})$ and $|m| \leq k$.

(i) $h(r)$ solves $-\frac{\hbar^2}{2} h''(r) + \frac{\hbar^2 l(l+1)}{2r^2} h(r) + V(r) \cdot h(r) = E \cdot h(r)$ ← ode in r L_3 -dep from L_-

(ii) $Y_{km}(\theta,\phi)$ are the spherical harmonics: $Y_{km}(\phi,\theta) = \frac{1}{\sqrt{2\pi}} e^{im\phi} P_n^m(\cos\theta)$ \leftarrow Legendre polynomials

* Eigenvalue is non-deg.

Q 2. Irreps for $SO(3)$: consider the space of harmonic polynomial with homog. degree k :

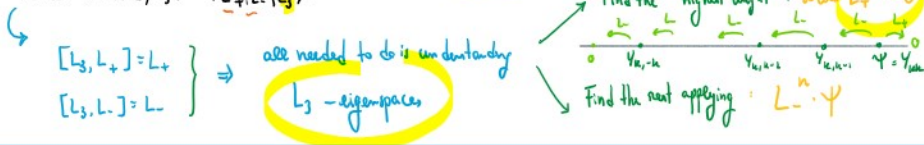
$$SO(3) \curvearrowright H^k := \{P \in \mathbb{C}[x,y,z] : P \text{ homog. of deg } k \text{ and } \Delta_{\mathbb{R}^3} P = 0\} \leftarrow E_x: \dim 2k+1.$$

because of homog. + "Δ metric" $\rightarrow \frac{(k+1)(k+2)}{2}$ describe the irrep.

Lemma: This is an irreducible rep. ← how do L_1, L_2, L_3 act?

Remember the FUNDAMENTAL COMPUTATION: Ψ eigenvect for M of eig. λ . $\Rightarrow A\Psi$ is then an eigenvect of M with eigenvalue $\lambda + \alpha$.
 Consider $L_{\pm} = L_1 \pm iL_2$ $[M, A] = \alpha \cdot A$

then $\langle L_1, L_2, L_3 \rangle = \langle L_{\pm}, L_{\pm}, L_3 \rangle$.



Q 1. The symmetries of central systems: q, p and angular momentum $q \times p$

Write $l = q \times p = (q_1 p_2 - p_1 q_2, q_2 p_3 - p_2 q_3, q_3 p_1 - p_3 q_1)$ quantities canonically $\hat{l} \cdot (L_1, L_2, L_3)$.

We can compute $[L_1, L_2] = iL_3$, $[L_2, L_3] = iL_1$, $[L_3, L_1] = iL_2$,

$[L_i, H] = 0$ because H central. \rightarrow what do they generate? Lie algebra by $L_1, L_2, L_3 + H$ piece.

Lemma: L_1, L_2, L_3 generate $so(3)$. $(e_1, e_2) = e_3, e_2, e_3 = iL_1$

Key question: How does $L^2(\mathbb{R}^3)$ decompose under $SO(3)$ symmetries?

(i) Peter-Weyl thm: any rep of $SO(3)$ is a direct sum of FINITE IRREDUCIBLE rep.

(ii) Irreps for $SO(3)$ are labeled by $k \in \mathbb{N}$: exactly 1 in $\dim 2k+1$. ($SO(3) \subset \mathbb{C}^{2k+1}$ irred.)

Q 3. Spherical harmonics: by defⁿ these are restriction of H^k , for some k , to S^2 .

In $(\theta, \phi) \in S^2$ we write $L_3 = -i\partial_\phi$, $L_{\pm} = e^{\pm i\phi}(\partial_\theta \pm i \cot\theta \partial_\phi)$ \leftarrow remember Ψ eig. of L_3
 Hence we need to solve: $L_{\pm} \Psi = 0 \Leftrightarrow \partial_\theta \Psi \pm i \cot\theta \partial_\phi \Psi = 0$ $\leftarrow -i\partial_\phi \Psi = k \cdot \Psi, k \in \mathbb{N}$
 so $\Psi_k = e^{ik\phi} \chi_k(\theta)$

$$\Leftrightarrow \partial_\theta \chi_k(\theta) - k \cot\theta \cdot \chi_k(\theta) = 0$$

This is solved by $\chi_k(\theta) \propto \sin^k \theta$. \Rightarrow highest weight eigenvect. is $Y_{kk}(\phi, \theta) = e^{ik\phi} \cdot \sin^k \theta$.

Then apply L_- to get $Y_{k,m} \leftarrow C \cdot e^{im\phi} \cdot \text{Poly.}(\sin\theta)$ \leftarrow labeled by $P_{mk}(\frac{z}{r})$.
 found by applying L_- : lower the poly.