Lecture 18

Tuesday, September 29, 2020 12:05 PM

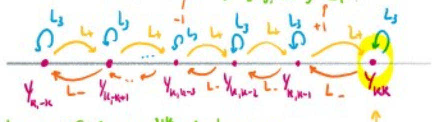
Lecture 18 : Free particle in S^2, hydrogen atom & other models

Monday : A problem in $\mathbb{R}^3$ with $SO(3)$ symmetry $V = V(r)$ breaks into $\rightsquigarrow$ $\left[-\frac{\hbar^2\Delta}{2m} + V(r)\right]\Psi = E\Psi$

(1) study of irreps for $SO(3)$

$so(3) = \langle L_1, L_2, L_3 \rangle = \langle L_-, L_+, L_3 \rangle$

then $[L_3, L_-] = -L_-$, $[L_3, L_+] = L_+$



$L_3 \rightsquigarrow k \in \mathbb{N}$ gives $\mathcal{H}^k$ of dim $2k+1$

$L_+ Y_{kk} = 0$

$\rightsquigarrow$ irreps labelled by $k \in \mathbb{N}$, each irrep of dim $2k+1$ has basis Y_{km}, $|m| \le k$ sph. harm.

(2) 1D radial problem of eigenvalues

operator. $h(r) = E \cdot h(r)$ 1-dim'l, $r \in \mathbb{R}$

this ODE might be hard: sol^n are not elem. fcts but also might have singularities

in 2010s "resurgence", WKB analysis

one idea: $r \in \mathbb{C}$ complexify.

§ 1. Free particle in S^2 : Since $V(r) = 0$, the stationary states are Ψ solving

$$\boxed{-\frac{\hbar^2 \Delta_{S^2}}{2m} \cdot \Psi = E\Psi}$$, note that $\Delta_{S^2} = -(L_1^2 + L_2^2 + L_3^2)$.

In fact $Y_{km}(\theta, \phi)$ is an eigenfunction : $\boxed{\Delta_{S^2} Y_{km} = -k(k+1) \cdot Y_{km}}$

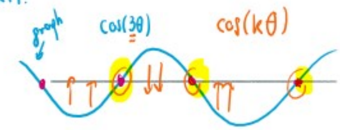
$\hookrightarrow$ the $E_k = -k(k+1)$ energy has eigenspace of dim $2k+1$.

Examples : $k=0$ $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$ has energy $E_0 = 0$

$k=1$
$$\begin{cases} Y_{1,\pm1}(\theta,\phi) = \mp\left(\frac{3}{8\pi}\right)^{1/2} \cdot e^{\pm i\phi} \sin\theta \\ Y_{1,0}(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cdot e^{0\cdot\phi i} \cdot \cos\theta \end{cases}$$

taking ∂_θ

$\rightsquigarrow$ as they evolve (solve Schrödinger) they do the same in S^2. VIDEO!



§ 2. Hydrogen atom : the potential is $V(r) = -C \cdot \frac{1}{r^2}$, C constant depends on charge, masses, etc. $C = -e^2 Z$.

Stationary states : at least $k > 0$, $|m| \le k$ but now more data needed because of 1D r-Schrödinger eq^n:

$$\boxed{h''(r) - \frac{k(k+1) - 2Zr - 2r^2E}{r^2} \cdot h(r) = 0}$$ $\rightsquigarrow$ sol^n exist if $E = E_n := \frac{-Z^2}{n^2}$, $0 \le k < n$

possible energies

the eigenspace is dim $\sum_{k=0}^{n-1} 2k+1 = n^2$

In conclusion, the stat. states are :

$$\boxed{\Psi_{nkm}(r,\theta,\phi) = \left[r^k e^{-Z/r} L_{nk}(r)\right] \cdot Y_{km}(\theta,\phi)}$$

$0 \le k < n$, $|m| \le k$

special family of poly. (Laguerre)

$\boxed{\text{3 labels } n, k, m}$

§ 3. Similar groups G : many groups $SU(1,1)$, $SO(4,2)$, $Sp(8,\mathbb{R})$ and more.

For the std. model : $U(1)$, $SU(2)$ & $SU(3)$

electromagn., weak, "same" as $SO(3)$ $\hookrightarrow$ labeled by k : $\mathcal{H}^k$

$\boxed{\text{what are the irreps ?}}$ $\rightarrow$ tables for irreps for $SU(N)$, $SO(N)$, $Sp(N)$

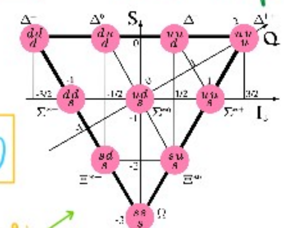
For $SU(3)$ the irreps are labeled by p, q, the space is called $D(p,q)$ w/ $\boxed{\dim D(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)}$

Rmk : $\dim SU(2) = 3$, $\dim SU(3) = 8$. $\rightsquigarrow$ play with 8 operators at once

$= \dim SO(3)$ playing w/ L_+, L_-, L_3

$S^5 \cong SU(3)/SU(2)$

e.g.



$\boxed{\text{example of } D(3,0), \ p = 3 \text{ quarks}, \ 10 \text{ dimensional}, \ q = 0 \text{ antiquarks}}$