COHERENT STATE

- quantum state obeying classical EoM

Application LASER

straight line trajectory

EXAMPLE: Harmonic Oscillator

$$H = \frac{1}{2}(p^2 + q^2)$$

$$= \hbar\left(N + \frac{1}{2}\right)$$

where $N = a^\dagger a$, $[a, a^\dagger] = 1$

Number operator

Vacuum $|0\rangle$ = Gaußian Wavepacket

Energy eigenstates

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

Obey $\langle n|m\rangle = \delta_{n,m}$

$$H|n\rangle = \hbar(n+\tfrac{1}{2})|n\rangle$$

Resolution of unity

$$\sum_n |n\rangle\langle n| = 1\!\!\mathrm{d}$$

$\mathcal{H}^*$ $\mathcal{H}$

Let $z \in \mathbb{C}$ and consider

$$|z\rangle := e^{za^\dagger}|0\rangle$$

which is an a-eigenstate b/c

$$a|z\rangle = [a, e^{za^\dagger}]|0\rangle = z e^{za^\dagger}|0\rangle = z|z\rangle$$

Unitary evolution

$$|z(t)\rangle := e^{-\frac{i}{\hbar}Ht}|z\rangle$$

$$= e^{-i(N+\frac{1}{2})t} e^{za^\dagger}|0\rangle$$

Lemma

$$e^{-iNt} f(a^\dagger) e^{iNt} = f(e^{-it} a^\dagger)$$

Proof $e^{-iNt} a^\dagger e^{iNt} =$

$Na^\dagger = a^\dagger(N+1)$

$$= a^+ e^{-i(N+1)t} e^{-iNt} = e^{-it} a^+ \quad \boxed{4}$$

Similarly

$$e^{-iNt} a^{+k} e^{iNt}$$

$$= \underbrace{e^{-iNt} a^+ e^{iNt} \cdots e^{-iNt} a^+ e^{iNt}}_{k\text{-times}}$$

$$= e^{-ikt} a^{+k} \qquad \square$$

Because $N|0\rangle = 0$,

$$e^{iNt}|0\rangle = |0\rangle$$

Hence

$$\boxed{|z(t)\rangle = e^{-\frac{it}{2}} |e^{-it} z\rangle}$$

$\Leftrightarrow$ coherent $\circlearrowright$ evolution

Classical motion $z = x + ip \Rightarrow$

$$x(t) = \mathrm{Re}\, z(t) = x\cos t + p\sin t$$

$$p(t) = \mathrm{Im}\, z(t) = p\cos t - x\sin t$$

Propagator

$$P_{z \xrightarrow[t]{} z'} = \left| \frac{\langle z' | z(t) \rangle}{\langle z' | z \rangle} \right|^2$$

The function

$$K(z, z'; t) := \langle z' | e^{-iHt/\hbar} | z \rangle$$

is called the (coherent state) propagator.

Observe

$$\frac{\partial}{\partial z} |z\rangle = \frac{\partial}{\partial z} e^{za^\dagger} |0\rangle = a^\dagger |z\rangle$$

$$z|z\rangle = [a, e^{za^\dagger}]|0\rangle = a|z\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}K = \langle z'|e^{-iHt/\hbar}H|z\rangle$$

$$= \langle z'|e^{-iHt/\hbar}(a^\dagger a + \tfrac{1}{2})|z\rangle$$

$$= \left(\frac{\partial}{\partial z}z + \frac{1}{2}\right)K$$

$\Rightarrow$ The propagator K obeys the Schrödinger equation

<u>Remark</u> In QFT the propagator $\langle out|e^{-\frac{iHt}{\hbar}}|in\rangle$ as $t \to \infty$ is called the <u>S-matrix</u>

Resolution of unity

Claim

$$\mathbb{1} = \int \frac{d\bar{z}dz}{2\pi i} |z\rangle e^{-\bar{z}z} \langle z|$$

Proof: $d\bar{z}\, d\bar{z} = 2i\, dx \wedge dy$

$$= 2i\, r dr\, d\theta$$

$$e^{-\bar{z}z} = e^{-(x^2+y^2)} = e^{-r^2}$$

$$|z\rangle\langle z| = \sum_{n,m} \frac{\bar{z}^n z^m}{\sqrt{m!n!}} |n\rangle\langle m|$$

$$= \sum_{n,m} \frac{r^{n+m} e^{i(m-n)\theta} |n\rangle\langle m|}{\sqrt{m!n!}}$$

$$\int_0^{2\pi} d\theta\, e^{i(m-n)\theta} = 2\pi\, \delta_{m,n}$$

Orchestrating

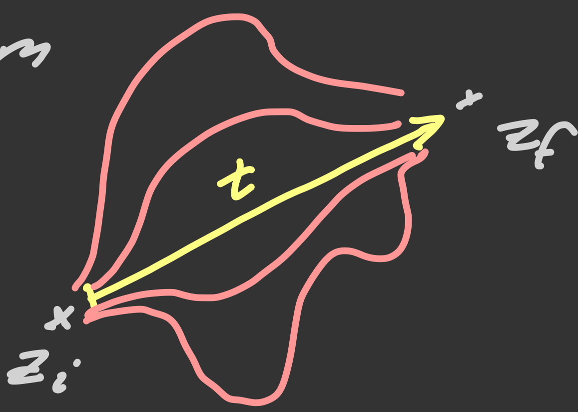
$$RHS = \frac{2\pi}{\pi} \sum_n \int \frac{r^{2n+1}e^{-r^2}}{n!} |n\rangle\langle n|$$

$$= \sum_n \underbrace{\int_0^\infty \frac{u^n}{n!} e^{-u}}_{1} |n\rangle\langle n|$$

$$= Id \qquad \square$$

Path integral

Claim



$\mathbb{C}$

Quantum propagator & hence probability is obtained by integrating over all paths connecting z & z' with a suitable measure.

Idea (Feynman), split large time quantum evolution into infinitesimal pieces & concatenate.

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right) = \underbrace{e^{-\frac{i\hat{H}\Delta t}{\hbar}} \cdots e^{-\frac{i\hat{H}\Delta t}{\hbar}}}_{N \text{ times}}$$

$$\Delta t := \frac{t}{N} \xrightarrow[N\to\infty]{} 0$$

Dirac: Small time evolution controlled by classical action principle! [L10]

$$\langle z' | e^{-\frac{i\hat{H}\Delta t}{\hbar}} | z \rangle$$

$$\approx \langle z' | \left(1 - \frac{i\hat{H}\Delta t}{\hbar}\right) | z \rangle$$

$$= \langle z' | \left(1 - \frac{i(a^\dagger a + \frac{1}{2})\Delta t}{\hbar}\right) | z \rangle$$

Harmonic oscillator

$$= \langle z' | z \rangle \left(1 - \frac{i(\bar{z}' z + \frac{1}{2})\Delta t}{\hbar}\right)$$

$$\approx \langle z' | z \rangle e^{-i(\bar{z}' z + \frac{1}{2})\Delta t}$$

exp("Liouville form")

classical Hamiltonian $p^2 + q^2$

Remark The relation

$$\langle z' | \hat{H} | z \rangle = H_{class}(z', z)$$

only holds for "free" theories

$\Rightarrow$ in general get

$$H_{class} + \mathcal{O}(\hbar)$$

$\uparrow$ "counter-terms"

$\rightsquigarrow$ cf. Quantum Field Theory

Trick Insert resolutions of unity to concatenate Dirac result

Recall $\mathbb{1} = \int \mu_i \, |z_i\rangle e^{-\bar{z}_i z_i} \langle z_i|$

so that

$K(z', z; t) =$

$$\int \mu_1 \cdots \mu_{N-1} \langle z | e^{\frac{-i\hat{H}\Delta t}{\hbar}} | z_{N-1}\rangle \, e^{-\bar{z}_{N-1} z_{N-1}} \langle z_{N-1} | \cdots$$

$[\mu]$

$$\langle z_i | e^{\frac{-i\hat{H}\Delta t}{\hbar}} | z_{i-1}\rangle \times e^{-\bar{z}_{i-1} z_{i-1}} \cdots$$

$$e^{-\bar{z}_1 z_1} \langle z_1 | e^{\frac{-i\hat{H}\Delta t}{\hbar}} | z\rangle$$

$$\approx \int [\mu] \langle z | z_{N-1}\rangle e^{-iH_{cl}(\bar{z}, z_{N-1}) - \bar{z}_{N-1} z_{N-1}}$$

$$\cdots e^{-\bar{z}_1 z_1} e^{\frac{-iH_{cl}(\bar{z}_1, z)\Delta t}{\hbar}} \langle z_1 | z\rangle$$

Note that

$$\langle z' | z \rangle = \langle 0 | e^{\bar{z}' a^{\dagger}} e^{z a} | 0 \rangle$$

$$= \sum_{m,n} \langle 0 | a^{\dagger m} a^{n} | 0 \rangle \times \frac{\bar{z}'^{m} \, z^{n}}{m! \; n!}$$

$$= \sum_{m,n} \delta_{m,n} \, m! \, \frac{\bar{z}'^{m} z^{n}}{m! \; n!}$$

$$= \exp(\bar{z}' z)$$

Orchestrating

$$K(z';z;t) \approx \int [\mu] \times$$

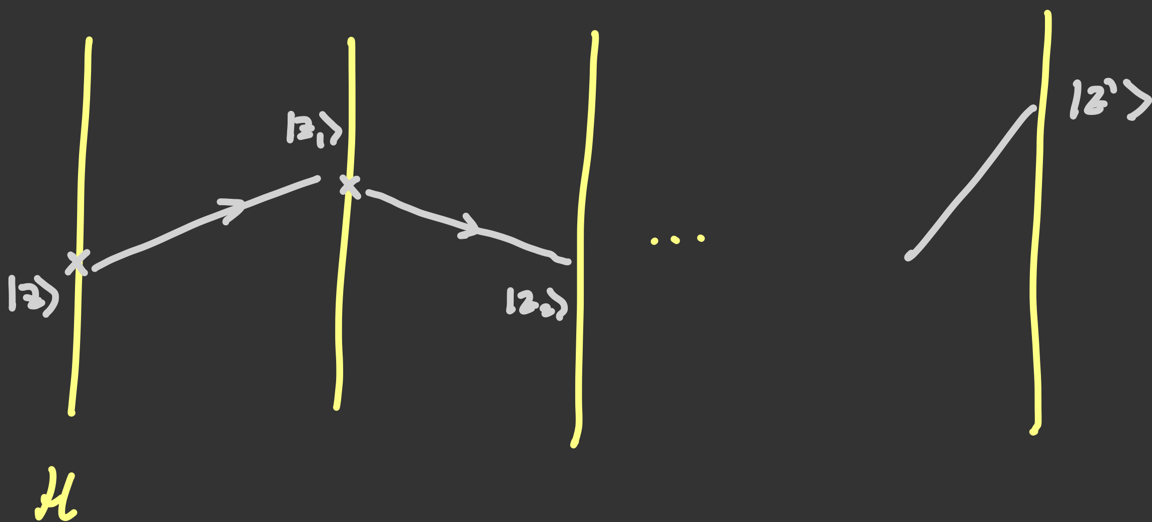
$$e^{(\bar{z} - \bar{z}_{N-1}) z - \frac{i}{\hbar} H_{cl}(\bar{z}, z_{N-1}) \Delta t}$$

$$\times\, e^{(\bar z_{N-1} - \bar z_{N-2}) z_{N-1} - \frac{i}{\hbar} H_{cl}(\bar z_{N-1}, z_{N-2})\Delta t}$$

$$\vdots$$

$$\times\, e^{(\bar z_2 - \bar z_1) z - \frac{i}{\hbar} H_{cl}(\bar z_1, z)}\; e^{\bar z_1 z}$$

A picture



$$\Delta t = \frac{t}{N}$$

As $N \longrightarrow \infty$

approximate

$$z_i \approx z(\tau) \in \mathbb{C}$$

$$\tau \in [0, t].$$

Also $(\bar{z}_i - \bar{z}_{i-1}) z_i = \frac{\bar{z}_i - \bar{z}_{i-1}}{\Delta t} z_i \Delta t$

$$\approx \dot{\bar{z}}(\tau) z(\tau) \, d\tau$$

Thus

$$\boxed{K(z', z; t) = e^{|z|^2} \int [\mu] \exp \frac{i}{\hbar} L(z, \bar{z})}$$

integration over paths from z to z'

PATH-INTEGRAL

Where

$$L = \int_0^t \left(\frac{\hbar}{i} \dot{\bar{z}} z - H_{cl}(z(\tau), \bar{z}(\tau)) \right) d\tau$$

What does an integration over paths mean?

How to compute path integrals?

SEMI-CLASSICAL APPROXIMATION

Given Lagrangian

$L(X(t))$ where

$X : I \longrightarrow M$

worldline

target space

We "define" the quantization of L by a path integral

$$K_{fi} := \langle X_f | e^{\frac{-i\hat{H}t}{\hbar}} | X_i \rangle = \int_{X_i}^{X_f} [dX] \, e^{\frac{iL[X]}{\hbar}}$$

paths from X_i to X_f

Suppose that $y(t)$ extremizes L.

i.e. $$\frac{d}{ds} L\big(y(t) + s\,z(t)\big)\Big|_{s=0} = 0$$

Then

$$L(y+z) = L(y) + O(y^2)$$

Thus

$$K_{fi} \approx e^{\frac{iL(y)}{\hbar}} \int [dz]\, e^{i[L(y+z) - L(z)]}$$

Dirac's leading approximation

Handle perturbatively

GRAPHICAL EXPANSION

Feynman Graphs