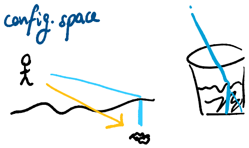
Lecture 2

Tuesday, September 29, 2020 12:05 PM

Lecture 2 : Lagrangian Formalism ← classical mechanics

$\Omega(a,b) := \{C^\infty\text{-maps } \gamma : [a,b] \longrightarrow M\}$ ← config. space



- $S : \Omega \longrightarrow \mathbb{R}$ action functional

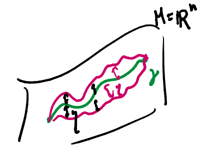
  $\gamma \longmapsto S(\gamma)$

- In this context, we'll start $M = \mathbb{R}^n_q$, $TM = \mathbb{R}^n_q \times \mathbb{R}^n_{\dot q}$ ← notation

  and $S(\gamma) = \int_a^b L(t, \underbrace{\gamma(t)}_{q}, \underbrace{\dot\gamma(t)}_{\dot q})\, dt$ → allowing $\ddot\gamma(t), \gamma^{(n)}(t)$ is fine

here the Lagrangian is $L : TM \times \mathbb{R}_t \longrightarrow \mathbb{R}$.

The principle of least action states that allowed trajectories are MINIMA of S. (In general, are $\mathrm{crit}(S)$.)



Skill to acquire : $\boxed{\text{FIND CRITICAL POINTS OF A FUNCTIONAL } S}$

$\delta S(\gamma)^{(\eta)} := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma + \varepsilon\cdot\eta)$ is the "partial derivative" in the direction of η, where $\eta \in \Omega$ with η fixed at endpoints. $(\eta(a) = \eta(b) = 0)$

Def: $\boxed{\text{A point } \gamma \in \Omega \text{ is critical if } \underbrace{\delta S(\gamma) = 0}_{\text{is a diff. eq.}}, \text{ for all } \eta.}$

Ex: In $M = \mathbb{R}^n$, $L(t, q, \dot q) = L(q, \dot q) = \underbrace{\frac{m}{2}\dot q^2}_{\text{kinetic energy}} - \underbrace{V(q)}_{\text{potential}}$.

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§ 2. Euler-Lagrange Eq's : sol^n of them are γ which are critical for $S(\gamma) = \int L$.

$\delta S(\gamma) = 0 \iff \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma + \varepsilon\eta) = 0 \iff 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_a^b L(t, \underbrace{\gamma + \varepsilon\eta}_{q}, \underbrace{\dot\gamma + \varepsilon\dot\eta}_{\dot q})\,dt$

Let's compute the ε-order of $\int_a^b L(t, \gamma+\varepsilon\eta, \dot\gamma + \varepsilon\dot\eta)\,dt$: we expand by Taylor

$$\int_a^b \Big[\partial_q L(t, \gamma+\varepsilon\eta, \dot\gamma+\varepsilon\dot\eta)\Big|_{\varepsilon=0} \cdot \eta + \partial_{\dot q} L(t, \gamma+\varepsilon\eta, \dot\gamma+\varepsilon\dot\eta)\Big|_{\varepsilon=0} \cdot \dot\eta\Big]\,dt. \quad (*)$$

Now $* = 0$ iff $\int_a^b \partial_q L(t,\gamma,\dot\gamma)\cdot\eta - \frac{d}{dt}\partial_{\dot q} L(t,\gamma,\dot\gamma)\cdot\eta\; dt + \cancel{\Big[\partial_{\dot q} L \cdot \eta\Big]_a^b} = 0$ ← bc $\eta(a) = 0$, $\eta(b) = 0$

iff $\int_a^b \left(\partial_q L - \frac{d}{dt}\partial_{\dot q} L\right)\eta\, dt = 0, \quad \forall \eta.$

$\gamma : (a,b) \longrightarrow M = \mathbb{R}^n_q$

$(\gamma, \dot\gamma) \overset{\exists}{\longrightarrow} TM = \mathbb{R}^n_q \times \mathbb{R}^n_{\dot q}$ $\quad q = \gamma,\ \dot q = \dot\gamma$

Hence $\delta S(\gamma) = 0$ iff $\boxed{\partial_q L(\gamma) - \frac{d}{dt}\partial_{\dot q} L \equiv 0}$. (#)

Def: (#) are known as the Euler-Lagrange associated to the action $S = \int L$. (It is generally a 2nd ode.)

Ex: $L(q, \dot q) = \frac{m}{2}\dot q^2 - V(q)$, then (#) read

$-\partial_q V - m\cdot\ddot q = 0$, equivalently

$\left\{\begin{array}{l} \partial_q L = -\partial_q V = -\nabla V. \\ \partial_{\dot q} L = m\cdot\dot q \\ \frac{d}{dt}\partial_{\dot q} L = m\cdot\ddot q \end{array}\right.$

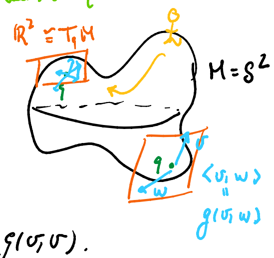
$m\cdot\ddot q = -\nabla V \quad \overset{F = -\nabla V}{\longleftrightarrow} \quad \boxed{m\cdot\ddot q = F}$.

§3. General configuration spaces : $TM \xrightarrow{\pi} M$ tangent bundle.

$\mathbb{R}^n \cong \pi^{-1}(q) \longrightarrow q \in M$ (dim $M = n$)

↳ possible velocities at q

(1) • The potential term is just $V \in C^\infty(M)$, which gives $V \circ \pi : TM \longrightarrow \mathbb{R}$.



(2) • The kinetic term is generalized by introducing a metric $g : TM \otimes TM \longrightarrow \mathbb{R}$.

From g we construct $g : TM \longrightarrow \mathbb{R}$ via $g(v) := g(v,v)$.

↳ the size of a tg. vector

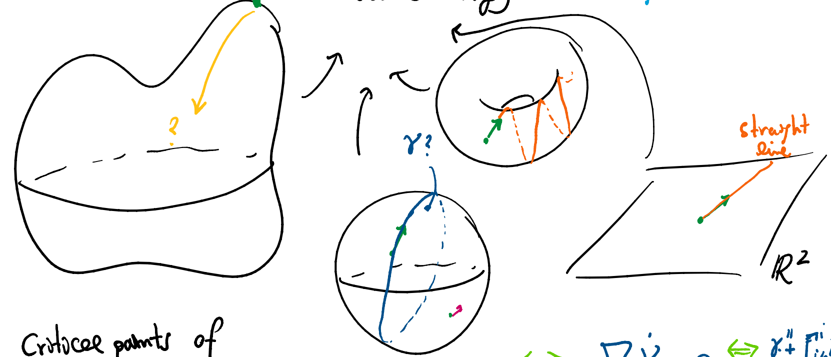
(3) • A Lagr. $L : TM \longrightarrow \mathbb{R}$ is typically $L = \frac{1}{2} g - V \circ \pi$.

mass is included in g.

kinetic — potential

Ex: The case of (M, g) with zero potential. → no force exerted to particle

GEODESICS in (M, g)



Prop: Critical points of $S(\gamma) = \int \frac{1}{2} g$ are geodesic paths. $\Longleftrightarrow \nabla_{\dot\gamma} \dot\gamma = 0 \Leftrightarrow \ddot\gamma^i + \Gamma^i_{jk} \dot\gamma^j \dot\gamma^k = 0$