Lecture 22

Tuesday, September 29, 2020 12:05 PM

Lecture 22 : CLASSICAL FIELD THEORY (introducing KG fields)

a section of a certain bundle

For now, we study $\mathbb{R}^3$ · space-time $(\mathbb{R}^3_q \times \mathbb{R}_t, g_{Mink})$ - $dt \otimes dt + \sum_{i=1}^{3} dq_i \otimes dq_i$

* configuration space is ∞-dim'l (diff. from clas. mech.)

Klein-Gordon thy: fields are scalar fields: $\boxed{\varphi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}}$, * it could be $\mathbb{C}$.

The action is $S(\varphi) := \int_{t_0}^{t_1} L(\varphi)\, dt$, where the Lagrangian is:

$$\boxed{L(\varphi) = \int_{\mathbb{R}^3_q} \underbrace{\frac{1}{2}|\partial_t \varphi|^2}_{\text{KINETIC}} - \Big(\underbrace{|\nabla \varphi|^2 + m^2 \varphi^2}_{\text{POTENTIAL}}\Big)}$$

$c = 1$ speed of light and $\hbar = 1$ as well.

$\partial_\varphi L - \frac{d}{dt}\partial_{\dot\varphi} L$

EULER-LAGRANGE

critical points satisfy:

$$\boxed{-\partial_t^2 \varphi + \nabla^2 \varphi = m^2 \varphi} \leftarrow \text{KLEIN-GORDON EQ}^n.$$

writing $\Box := -\partial_t^2 + \nabla^2 = \Delta^2_{Mink}$ we get

$$\boxed{\Box \varphi = m^2 \varphi}$$

§1. Solving KG eq^n:

Being linear, we solve KG via Fourier:

$$\varphi(q,t) \overset{\text{Fourier decomp.}}{=} \int_{\mathbb{R}^3_q} \hat{\varphi}(\vec{p},t) \cdot e^{i\vec{p}\vec{q}}\, d\vec{p}, \quad \text{and note } \hat{\varphi}(-p,t) = \hat{\varphi}(p,t)^* .$$

φ being $\mathbb{R}$.

plug into KG

$$(-\partial_t^2 + \nabla^2 - m^2)\varphi = 0 \iff \int_{\mathbb{R}^3_q} e^{ip\cdot q}\left(-\partial_t^2 - |p|^2 - m^2\right)\varphi(p,t)\, dp .$$

$\Rightarrow$ {the lesson is that for a fixed p, we get an HARMONIC OSCILLATOR}

$$\boxed{\ddot{\varphi}(p,t) + (|p|^2 + m^2)\varphi(p,t) = 0}$$

h.o. with freq. $\omega(p) = |p|^2 + m^2$

$$\Rightarrow \varphi(q,t) = \int_{\mathbb{R}^3} a(p)\cdot e^{i(p\cdot q - \omega(p)t)} + a(p)^* e^{-i(p\cdot q + \omega(p)t)}\, dp$$

§2. What about quantizing?

π is the "conjugate of φ": for scalar field this $\partial_t \varphi =: \pi$.

Option 1: Consider the Hamiltonian

$$\boxed{H(\varphi) := \int_{\mathbb{R}^3} \frac{1}{2}|\pi|^2 + |\nabla \varphi|^2 + m^2 \varphi^2\, dq .}$$

KG Hamiltonian

$\leftarrow$ quantizing H involves quantizing φ itself "2nd quantization".

Option 2: Quantize via path integral

i.e. $Z(J) = \int e^{iS} = \int e^{i\int L + J\cdot\varphi}$ } $\rightsquigarrow$ obtains directly propagators K $\rightsquigarrow$ use it to obtain $\Psi(q_f, t_f)$ from $\Psi(q_i, t_i)$.

Quantization via H:

conf. space $\xrightarrow{\text{quantized}}$ $\mathcal{H}$ = Hilbert space $:=$ "$L^2(\text{conf. space})$" = wave functionals

observables $\xrightarrow{\text{quantized}}$ operators on this $\mathcal{H}$.

In KG field theory, what are observables?

(i) Given $q \in \mathbb{R}^3$, $ev_q(\varphi) = \varphi(q)$ is the classical field analogue of position.

(ii) In general, given $h \in C^\infty(\mathbb{R}^3, \mathbb{R})$, $\varphi \longmapsto \int_{\mathbb{R}^3} \varphi(q)\cdot h(q)\, dq$.

(iii) The classical mech. momentum p gets sent to the derivative of the field.

classical field theory observables