Lecture 23

Tuesday, September 29, 2020 12:05 PM

Lecture 23 : Quantization of Klein-Gordon

$\mathcal{L}_{KG}, H_{KG} \rightsquigarrow (\underbrace{-\partial_t^2 + \nabla^2}_{\Box} + m^2)\varphi = 0$ "wave eq." $(\hbar = 1, c = 1)$

$$\varphi(q,t) = \frac{1}{(2\pi)^3}\int \frac{1}{\omega(p)}\left(a(p)\, e^{-ipq} + a^*(p)\, e^{ipq}\right) d^3p.$$

analog of q; $\omega(p) = (p^2+m^2)^{1/2}$; 4-vect. in Mink $(-1,1,1,1)$; a, a^*: Fourier coeff.

analog of p:

$$\pi(q,t) = \frac{\delta L_{KG}}{\delta \partial_t \varphi} = \partial_t \varphi = \frac{i}{2(2\pi)^3}\int -a(p)\, e^{-ipq} + a^*(p)\, e^{ipq}$$

} classical field thy.

Hilb. space will "WAVEFUNCTIONALS"

$\Psi_0(\varphi) \in \mathbb{C}$, φ field.

upgrade ϕ & π to operators : $[\hat{\varphi}(q,t), \hat{\varphi}(y,t)] = 0$, similarly $[\hat{\pi}(q,t), \hat{\pi}(y,t)] = 0$

and $[\hat{\varphi}(q,t), \hat{\pi}(y,t)] = i\,\delta^{(3)}(q-y)$. ← Dirac ; generalize Heisenberg

$\boxed{\text{No STONE-VON NEUMANN THM: non-uniqueness!}}$

Following the a, a^* from the Harm. Oscill. : $\hat{a}(p), \hat{a}(p)^*$ satisfy

$[a(p), a(k)] = 0$

$[a^*(p), a^*(k)] = 0$

$+\ [a(p), a^*(k)] = \delta(p-k)\cdot \text{Cont}\cdot\omega(p)$ } $\Rightarrow$ $\boxed{\hat{H} = \frac{1}{2}\int_{\mathbb{R}^3_p} \frac{1}{(2\pi)^3}(\hat{a}\hat{a}^* + \hat{a}^*\hat{a})\, d^3p \quad \text{"vacuum has } \infty E\text{"}}$

§ 2. Vacuum of KG :

following $H \sim aa^* + a^*a$, vacuum given by $\boxed{\hat{a}\cdot\Psi_0 = 0}$.

Prop: The vacuum wavefunctional is

$$\Psi_0(\varphi) = e^{-\int_{\mathbb{R}^3_p}\int_{\mathbb{R}^3_q}\int_{\mathbb{R}^3_y} \frac{\omega(p)}{2(2\pi)^3}\cdot e^{i(q-y)\cdot p}\,\phi(q)\phi(y)\, dy\, dq\, dp}.$$

no particles!

Start playing with Ψ_0 : $\hat{a}^*(p)\cdot\Psi_0$, $\widehat{ev}_q\cdot\Psi_0$, $(\hat{a}^*(p_1)\,\hat{a}(p_2)^*\,\widehat{ev}_{q_1}\,\widehat{ev}_{q_2})\Psi_0$

4 particles created from KG vacuum.

(i) $\hat{a}^*(p)$: creates a particle with momentum p

(ii) $\widehat{ev}_q$: creates a particle at q

$\rightsquigarrow$ $\boxed{\text{PARTICLES in QFT are "field excitation".}}$

Properties & Generalizations

(1) The Hilbert space is the "Fock space" $= \bigoplus_{k\geq 0} H^{\otimes k}$ (1 particle: H), we use a symmetric version $\rightsquigarrow$ bosons

(2) The vacuum energy is infinite : to fix this do "normal ordering", apply aa^* always a first.

$:H: \;= \frac{1}{2}:aa^* + a^*a: \;= a^*a \rightsquigarrow\; :H:\Psi_0 = a^*a\Psi_0 = 0.$

Another ∞ appears in normalizing $\|\Psi_0\| = \infty$.

(3) Complex scalar field : $\phi(q) \sim \int a(p)\, e^{ipq} + b(p)\, e^{-ipq}$. $\Rightarrow$ 2 sets of particles a, a^* b, b^*

Really, symmetry $\phi \mapsto e^{i\theta}\phi$. $\xrightarrow{\text{Noether}}$ $\hat{Q}$ a charge $\rightsquigarrow$ a, a^* eigenv. 1 — positive charge; b, b^* eigenv. -1 — neg. charge } particles, anti-particle

(4) Dirac field : "$\sqrt{\Box + m^2}$" $\varphi = 0$? $\rightsquigarrow$ matrix valued fields, "Pauli matrices" $\rightsquigarrow$ $\boxed{\text{anticommutation relations}}$ $\rightsquigarrow$ FERMIONS

§ 3. Path Integral Quantization :

L Lagrangian, $L = L_{KG} = -\partial_t^2 - \nabla^2 - m^2$

The propagators are given by oscill. integral : vacuum to vacuum (in $-\infty$ to ∞).

also $Z[J] = e^{w[J]}$ (*)

$$Z[J] = \int e^{i\int_{t_i=-\infty}^{t_f=\infty} L + J\cdot\varphi}$$ "partition" fct

gets us $\langle 0|\, \hat{\phi}(q_1)\hat{\phi}(q_2)\dots\hat{\phi}(q_n)\, |0\rangle^J = \left.\frac{\delta Z[J]}{\delta J(q_1)\dots\delta J(q_n)}\right|_{J=0}$ (time-ordered; vacuum wavefunct.)

$\partial(\varphi\partial\varphi) = \partial\varphi^2 + \varphi\cdot\partial^2\varphi$

$$Z[J] = \int e^{i\int \frac{1}{2}|\partial\varphi|^2 - m^2\phi^2 + J\phi} = \int e^{i\int -\frac{1}{2}\varphi\cdot\partial^2\varphi - m^2\varphi + J\varphi} = \int e^{-i\int_{\mathbb{R}^4} \underset{q_i}{\varphi}\cdot\underbrace{(\partial_t^2+m^2)}_{K}\underset{q_i}{\varphi} + \underset{J}{J}\cdot\varphi} =$$

$$= \int e^{-i\iint_{\mathbb{R}^4_q\mathbb{R}^4_y}\varphi(q)\,(\partial_t^2+m^2)\cdot\delta^{(4)}(q-y)\,\varphi(y) + J\cdot\varphi} = \frac{1}{\det(K)^{1/2}}\, e^{\frac{1}{2}\int J(q)\cdot\Delta_F\, J(y)}.$$ (*)

$$\boxed{\int_{-\infty}^{\infty}\prod e^{-i/2\cdot\sum \varphi_i K_{ij}\varphi_j + \sum J_i\varphi_i} = (2\pi)^{n/2}\cdot\det(K)^{-1/2}\cdot e^{\frac{1}{2}\sum J_i (K^{-1})_{ij} J_j}}$$ (K symmetric)

(*) $\Delta_F(q,y) := [\delta(q-y)(\partial_t^2+m^2)]^{-1}$ Green's fct for Klein-Gordon

$$\Delta_F(q,y) = \int_{\mathbb{R}^4}\frac{1}{p^2-m^2}\, e^{-ip(q-y)}\, dp$$ $\rightsquigarrow$ Feynman's propagator