Lecture 24

Tuesday, September 29, 2020 12:05 PM

Lecture 24 : Yang-Mills Field Theory (gauge theory) : classical, its crucial for the std. model, quantization still open.

Ingredients : (M^4, g) Riem. 4-fold., G a Lie gp. (abelian / non-ab.)

→ a principal G-bundle, A connection (1-form, gauge potential), F_A curvature (2-form valued in matrices, gauge field).

- $G = U(1)$ electromag. → quant. QED
- $G = U(1) \times SU(2)$ electroweak ← deal with mass : Higgs field
- $G = U(1) \times SU(2) \times SU(3)$ → strong nuclear force "asympt. freedom"
  - $SU(3)$ → QCD ↳ cromodynamics: rep of $SU(3)$ are labeled by "quarks".

Def: The Yang-Mills functional is :

$$L_{YM}(A) := \int_{M^4} tr\left(F_A \wedge ({*}_g^{(1)} F_A)\right) d\mu_{vol(M)} \in \mathbb{R}.$$

A: a connection on a G-ppal bundle

Euler Lagrange give eq^n of motions

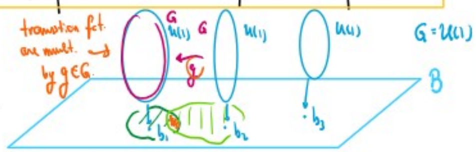
The Yang-Mills eq^n are : $d_A({*}_g F_A) = 0$ and $d_A F_A = 0$

always true : Bianchi identity

(1) $*(dx_{i_1} dx_{i_2} \cdots dx_{i_k}) = \pm dx_1 dx_2 \cdots \widehat{dx_{i_1}} \cdots dx_n$

§ 2. Principal bundles : G a Lie group and B (base) a smooth manifold. ($B = M^4$ for us.)

Def: A principal G-bundle is a smooth manifold P and a map $\pi : P \longrightarrow B$ (π is proj. here) such that π is locally trivial, i.e. $\forall b \in B$ $\exists G_p(b) \subseteq B$ s.t. $\pi^{-1}(G_p(b)) \cong G_p(b) \times G$.



Rmk: The classif. of ppal G-bundle depends solely on alg top. : $H^*(B; \pi_*(G))$ tell you enough.

{ E.g. $\pi_*(U(1)) = \pi_1(U(1)) = \mathbb{Z}$ for $* \geq 1$, then $U(1)$-bundles on S^2 are given by $H^2(S^2; \pi_1(U(1))) \cong \mathbb{Z}$. } ⇝ take MAT215 and you can compute.

(1) P has a section iff $P = B \times G$. (different from v.b. !)

(2) If $\rho : G \to GL(V)$ is a rep., we can create a v.b. "associated" bundles ⇝ $P \times_\rho V$ "substituting the G fibers by V".

every v.b. is an associated bundle !

§ 3. Parallel transport & Connections : how to identify two fibers ?

$U(1) \cong S^1$, $SU(2) \cong S^3$

Parallel transport : assign to each path γ from $b_1, b_2 \in B$ an isomorphism $PT_\gamma : G \longrightarrow G$, with $G = \pi^{-1}(b_1)$ and $G = \pi^{-1}(b_2)$.

PT_γ might depend on γ, not just htpy type.

(same with v.b.).

integrated ↑ ↓ derived version

Connection : the idea is to give data (a $\mathfrak{g}$-valued 1-form) such that PT_γ is given by solving an ODE for this data.

$A \in \Omega^1(P; \mathfrak{g})$ ⤳ to be $\mathfrak{g}$-valued is being a section of $T^*P \otimes \mathfrak{g} \longrightarrow P$: in practice A a matrix-valued 1-form, equiv. a matrix of 1-form.

$\mathfrak{g}$ ↪ End(G)

matrix ≅ elt. of $\mathfrak{g}$.

s section $s : B \to P$

The covariant derivative for A is

$$d_A := d + A$$

$s : B \to P$ sect.
$d_A s : TB \to TP \otimes \mathfrak{g}$
$ds : TB \to TP$
$s \mapsto ds + As$

⇝ PT_γ is obtained by solving ODE $d_A s = 0$. ← "flat sections"