Lecture 25

Tuesday, September 29, 2020 12:05 PM

Lecture 25: Connections, gauge & curvatures

P a ppal G-bundle

(i) $\pi : P \xrightarrow{G \text{ group}} B = P/G$ — G acts freely on P fibrewise

p a rep of G ⇝ $\pi : E \xrightarrow[\text{v.b.}]{\text{fibre is } V} B$ on some base

(ii) Connection on P: $\Omega^1(P; ad(G))$ (adjoint on $\mathfrak{g}$) ⇝ Connection on assoc. bundle $E \longrightarrow B$ ⇝ horizontal complement

$G \circlearrowright \eta \in \Omega^1(P; \mathfrak{g})$

covariant deriv.: $d_A := d + A$ (locally; End-valued 1-form)

1-form (integrability to parallel transp)

(iii) Curvature of a connection ⇝ (s sect. s.t. $d_A s = 0$ is flat)

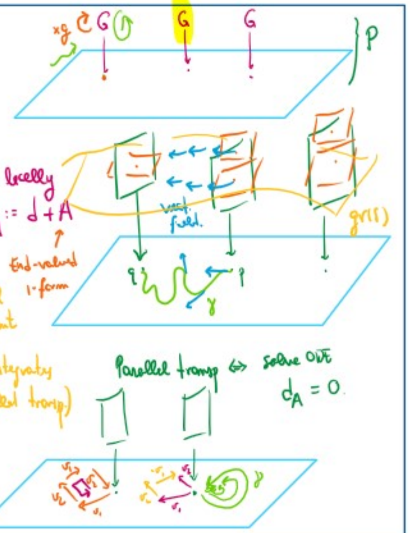
$G \circlearrowright \omega \in \Omega^2(P; ad(\mathfrak{g}))$.

G gauge group: quotient by this action!

$v_1, v_2 \to \mathrm{End}(\text{fibre})$ ⇜ $v_1, v_2 \in TB$ gets us a monodromy

2-form on B End-valued!

Parallel transp ⟺ solve ODE $d_A = 0$.

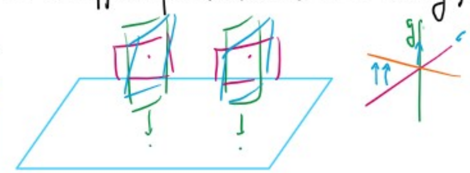


§2. Space of connections:

Lemma: the space of connections in P is an affine space, modelled on $\Omega^1(ad(\mathfrak{g}))$.

In particular it is contractible:

$\boxed{\text{topologically, the "moduli of connections" is } \simeq \text{pt.}}$



(The key pt. is that we need to quotient this space $\mathcal{A}$ (∞-dim'l affine space) by the action of $\mathcal{G}$, gauge group.

$d + A$ (connection) ⇝ $g \cdot (d + A)$ — g acts non-trivially in the A factor (ad-action)

⇝ the moduli space in Yang-Mills field theory are all connections (+ PDE for E-L) MODULO the gauge group!

§3. Yang-Mills: $L_{YM}(A) := \int_{M^4} \mathrm{tr}(F_A \wedge *_g F_A)$.

A is a connection $\in \Omega^1(P; ad(\mathfrak{g}))$; F_A curvature; $*_g F_A$ 2-form

tr becomes $F_A \wedge *F_A \in \Omega^4(P; \mathfrak{g})$

Kinetic energy: "$\|F_A\|^2 = \|\dot{A}\|^2$".

Euler-Lagr. eq^n: $L_{YM}(A + \varepsilon\eta) - L_{YM}(A) =$

$$= \int F_{A+\varepsilon\eta} \wedge *F_{A+\varepsilon\eta} - F_A \wedge *F_A \overset{(*)}{=} \varepsilon \int F_A \wedge * d\eta + O(\varepsilon^2)$$

$(*)$ $F_\eta := (d + \eta)^2 = d\eta + [\eta, \eta]$

Thus, critical points satisfy $\int F_A \wedge * d\eta = 0 \Leftrightarrow \int *F_A \wedge d\eta = 0 \Leftrightarrow \int d * F \wedge \eta = 0$.

$$\Rightarrow \boxed{d_A * F_A = 0}$$

(also, automatically $d_A F_A = 0$).

Rmk: If A is anti-self dual (ASD), i.e. $*F_A = -F_A$, then crit. point (abs. min.) of YM. ▫