Lecture 27

Tuesday, September 29, 2020 12:05 PM

Lecture 27 : Path integral quantization in gauge theory

— Yang Mills : QED, QCD, ??? — mass gap

— Kapustin - Witten

— Chern-Simons & more.

$G = U(1), SU(2)$.

classical field thy $\mathcal{L}$ (Lagr. (action) den) : $\mathcal{L}_{YM}, \mathcal{L}_{CS}, S_{cs} = \int \mathcal{L}_{CS}$

gauge fields : connections ⇝ modulo gauge G so as to count for physical conf.

→ classical space of vacua are interesting moduli spaces

Path integral quantization : $Z[J] := \int e^{i \cdot S + J \cdot A} \mu_{fields}$, A field. $S = \int_{x_i}^{x_f} \mathcal{L}$

(i) Integral is over all config., not just physical conf. : G ⊂ space of field

↳ NEED TO PICK 1 representative per orbit

Fadeev Popov ghost!

"gauge fixing"

(ii) The integral is oscillatory : try ex. $\int_{\mathbb{R}} e^{if(x)} \cdot a(x)\,dx$? (Fresnel integral) ⇝ (*) $\boxed{\text{the limit } \lim_{\lambda\to\infty} \int e^{i\lambda f} \cdot a \text{ can be computed !}}$

§ 2. Oscillatory integrals :

$I(\lambda) := \int_{\mathbb{R}^n} a(x) \cdot e^{i\lambda f(x)}\,dx$. phase, understand $\lim_{\lambda\to\infty} I(\lambda)$?

comp. supp "amplitude"

Fact : (Stationary phase pple.) (A_1) f is Morse, $G(\lambda^{-n/2})$

$$\int_{\mathbb{R}} a(x) \cdot e^{if(x)\cdot\lambda}\,dx = \sum_{x_0 \text{ crit. pt. phase}} a(x_0) \cdot e^{if(x_0) + \mu} \cdot \frac{\sqrt{2\pi}}{\sqrt{\lambda \cdot |f''(x_0)|}} + O(1/\sqrt{\lambda}).$$

μ ← maslov index (sign of f'')

det(Hess) for $n \in \mathbb{N}$ in $\mathbb{R}^n$

Ex : $\int e^{ix^3}dx$ (Airy) — A_2-sing

In QFT reality : $\mathcal{L}_{cs}(A) = \int_{M^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$,

$$\lim_{\lambda\to\infty} Z_\lambda[J] = \int e^{i\lambda \int \mathcal{L}_{cs}} \mu_{fields} = \sum e^{iS_{cs}(A_{flat})} \cdot \boxed{\text{"the determinant contribution"}}$$

$A_{flat} + B$

Ray-Singer analytic torsion

→ "δΔδ"

$\boxed{A \text{ s.t. } F_A = 0, \text{ i.e. } dA + A \wedge A = 0}$ moduli of flat connection ⇜ critical points of S_{cs}

Chern-Simons is topological (TQFT) — no choice of metric for S_{cs}

§ 3. Fadeev-Popov Regularization : ← 1 way to renormalize a QFT

The goal is to fix gauge repr. in an orbit : suppose the gauge condition is $\boxed{G(A) = 0}$.

(i) $Id \underset{(*)}{=} \det\left(\frac{\delta G(A_\alpha)}{\delta\alpha}\right)\Big|_{\alpha=0} \cdot \int \delta(G(A))\,d\alpha$, (delta) $\int_{\mathcal{A}} e^{iS_{cs}} \mu_{fields}$ "=" $\int_{\mathcal{A}/G} e^{iS_{cs}} \cdot \left(\int_{\text{each orbit}}\right)$

can be infinite, may remove it!

(*) $\delta(g(x)) = \sum_{g(x_0)=0} \frac{\delta(x-x_0)}{|g'(x_0)|}$, thus $\boxed{1 = |g'(x_0)| \cdot \int \delta(g(x))\,dx}$

(ii) What is the determinant ? The Berezin integral allows one to compute integrals in θ-vars,

θ is exterior algebra variable :

$$\boxed{\det M = \int e^{-\theta^t \cdot M \cdot \eta}\,d\theta\,d\eta}$$

matrix; fermionic (physics); anticommuting θ, η.

$\det(M) = \int \left(\prod d\theta_i d\eta_i\right) e^{\sum \theta_i M \eta_i}$