Lecture 4

Tuesday, September 29, 2020 12:05 PM

Lecture 4 : Noether's Principle

← suppose the system has symmetry

$S(\gamma) = \int L(\gamma, \dot\gamma, t)\, dt$ (Lagr.) $\xleftrightarrow[\text{(when possible)}]{\text{leg. transf}}$ $H : T^*M \to \mathbb{R}$ (Ham. formalism)

$L : TM \to \mathbb{R}$ $\qquad$ X_H Hamil v.f.

Q: what if L is infinitesimally invariant ? (See Friday "H has a symmetry.")

A: Then trajectories γ which are S-critical have constant of motion.

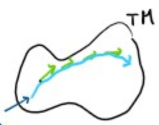
this is a function $f : M \to \mathbb{R}$ s.t. $f|_{\gamma} \equiv$ constant on traj.

the flow g_η^t of η will preserve L, i.e. $(g_\eta^t)^* L = L$.

$Y \rightsquigarrow \eta$ acts on any tensor of TM

Def: Given $\eta \in \mathfrak{X}(M)$ a vect. field, $L : TM \to \mathbb{R}$ is said to be infinitesimally inv't under η if $\underbrace{\mathcal{L}_\eta L}_{dL(\eta)} \equiv 0$.

bc v.f.



Thm: (Noether) Let $\tilde\eta$ (v.f.) be an infinitesimal symm. of L; ← $L : TM \to \mathbb{R}$

then $\langle \tilde\eta, -i_L^*(\lambda_{st}) \rangle$ is constant on a critical traj. of $\underbrace{S(\gamma)}_{\text{action fct}^{l.}}$.

↳ function on TM

(1). In the statement : $\eta \in \Gamma(TM)$ and we lift it to $\tilde\eta \in \Gamma(T(TM))$ a v.f. of TM.

locally $\eta = \sum_{i=1}^n \eta_i \partial_{q_i}$ gives $\tilde\eta := \sum_{i=1}^n \eta_i \partial_{q_i} + \sum_{j=1}^{n} \dot q_j (\partial_{q_j} \eta_i) \partial_{\dot q_i} \in \Gamma(T(TM))$ v.f. in TM

Being a symmetry is $\mathcal{L}_{\tilde\eta} L \equiv \boxed{dL(\tilde\eta) = 0}$

(2). Recall Euler-Lagr: $\boxed{\partial_{q_i} L = \frac{d}{dt} \partial_{\dot q_i} L}$

(3) The Legendre transform $i_L : TM \to T^*M$

$-i_L^*(\underbrace{\lambda_{st}}_{\sum_{i=1}^n p_i dq_i}) = \underbrace{\sum_{i=1}^n (\partial_{\dot q_i} L) \cdot dq_i}_{\text{1-form in TM}}$

Example 1: (Conservation of energy). $L = \frac{1}{2} \|\dot q\|^2 - V(q)$

Then $\frac{d}{dt}(L) = \frac{d}{dt} q_i \, \partial_{q_i} L + \frac{d}{dt} \dot q_i \cdot \partial_{\dot q_i} L \overset{\text{E-L holds}}{=} \left(\frac{d}{dt} q_i\right) \cdot \partial_{q_i} L + \dot q_i \frac{d}{dt} \partial_{\dot q_i} L$

$= \dot q_i$ $\qquad$ $\partial_q L \,\|\, \text{EL}$

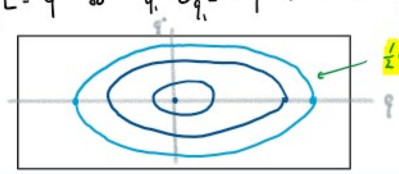
$\gamma : [0,1] \to M$ $\quad$ restrict L to γ and denote $d\gamma^*(L)$

$= \frac{d}{dt}(\dot q_i \, \partial_{\dot q_i} L) \implies \dot q_i \cdot \partial_{\dot q_i} L - L$ is conserved.

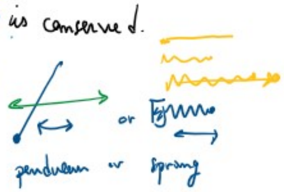
Application: $L = \frac{1}{2} \dot q^2 - q^2$ the 1-dim harmonic oscillator.

Since $\partial_{\dot q_i} L = \dot q$ so $\dot q_i \cdot \partial_{\dot q_i} L = \dot q^2$, hence $\frac{1}{2} \dot q^2 + q^2$ is conserved.

(phase portrait)



$\frac{1}{2}\dot q^2 + q^2 \equiv$ constant

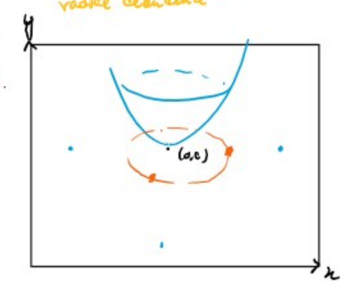


Example 2: Suppose $M = \mathbb{R}^2$, $L = \frac{m}{2}(\dot x^2 + \dot y^2) - V(\underbrace{(x^2+y^2)^{1/2}}_{\text{radial dependence}})$ $\quad$ ∂_ϕ inv.

By using (r, ϕ) : $L = \frac{m}{2} \dot r^2 (1 + \dot\phi^2) - V(r)$.

Exercise: $\partial_\phi L \equiv 0$ (dL has no $d\phi$, so $dL(\partial_\phi) \equiv 0$).

we compute $\langle -i_L^*(\lambda_{st}), \tilde\eta \rangle = \boxed{\frac{1}{2} m r^2 \dot\phi}$.



Application: if $V(r) \equiv 0$, then we have

$\frac{m}{2} \dot r^2$ is preserved (energy) (by Example 1) and also $\frac{1}{2} m r^2 \dot\phi$.

$\implies$ any trajectory will have to be a line. ∎